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Scaling limits of random walks and their related parameters on critical random trees and graphs

by

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Contents

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Declarations

I declare that, to the best of my knowledge, the content in this thesis is original and is formed by my own work, conducted under the supervision of Dr. David Croydon, except otherwise stated. Specifically,

i) Chapter 3 contains material from Andriopoulos [9];

ii) Chapter 4 contains material from Andriopoulos [9];

iii) Chapter 5 contains material from Andriopoulos [10];

iv) Chapter 6 contains a list of possible future working plans on open problems that can be investigated from the actual thesis;

v) Chapter 2 provides the necessary background and several technical results that we use throughout the articles noted above.

This thesis is submitted to the University of Warwick for the degree of Doctor of Philosophy and has not been submitted to any other higher education institution or for any other degree.

Abstract

In this thesis we study random walks in random environments, a major area in Probability theory. Within this broad topic, we are mainly focused in studying scaling limits of random walks on random graphs at criticality, that is precicely when we witness the emergence of a giant component that has size proportional to the number of vertices of the graph. Critical random graphs of interest include critical Galton-Watson trees and maximal components that belong to the Erdős-Rényi universality class.

The first part of the thesis expands upon using analytic and geometric properties of those random graphs to establish distributional convergence of certain graph parameters, such as the blanket time. Our contribution refines the previous existing results on the order of the mean blanket time. The study of this problem can be seen as a stepping stone to deal with the more delicate problem of establishing convergence in distribution of the rescaled cover times of the discrete-time walks in each of the applications of our main result.

Relying on powerful resistance techniques developed in recent years, another part of the thesis investigates random walks in random enviroments on tree-like spaces and their scaling limits in a certain regime, that is when the potential of the random walk in random environment converges. Results include novel scaling continuum limits of a biased random walk on large critical branching random walk and a self-reinforced discrete process on size-conditioned critical Galton-Watson trees. In both cases the diffusions that are not on natural scale are identified as Brownian motions on a continuum random fractal tree with its natural metric

replaced by a distorted resistance metric.

Chapter 1

Introduction

A simple random walk on a finite connected graph G with at least two vertices is a reversible Markov chain that starts at some fixed vertex, and at each step moves with equal probability to one of the vertices adjacent to its present position. The mixing and the cover time of the random walk are among the graph parameters which have been extensively studied. The mixing time measures the time required such that the distribution of the Markov chain is within small maximal total variation distance from the unique invariant measure. To these parameters, Winkler and Zuckerman [103] added the ε-blanket time variable (an exact definition will be given later in (3.4)) as the least time such that the walk has spent at every vertex at least an ε fraction of time as much as expected at stationarity. Then, the ε-blanket time of G is defined as the expected ε-blanket time variable maximized over the starting vertex.

The necessity of introducing and studying the blanket time arises mainly from applications in computer science. For example, suppose that a limited access to a source of information is randomly transferred from (authorized) user to user in a network. How long does it take for each user to own the information for as long as it is supposed to? To answer this question under the assumption that each user has to be active processing the information equally often involves the consideration of the blanket time. To a broader extent, viewing the internet as a (directed) graph, where every edge represents a link, a web surfer can be regarded as a walker who visits and records the sites at random. In a procedure that resembles Google's PageRank (PR), one wishes to rank a website according to the amount of time such walkers spend on it. A way to produce such an estimate is to rank the website according to the number of visits. The blanket time is the first time at which we expect this estimate to become relatively accurate.

Obviously, for every $\varepsilon \in (0,1)$, the ε-blanket time is larger than the cover time since one has to wait for all the vertices to have been visited at least once. Winkler and Zuckerman [103] made the conjecture that, for every $\varepsilon \in (0,1)$, the ε-blanket time and the cover time are equivalent up to universal constants that depend only on ε and not on the particular underlying graph G. This conjecture was resolved by Ding, Lee and Peres [45] (an exact statement will be given later in (3.5)) who provided a highly non-trivial connection between those graph parameters and the discrete Gaussian free field (GFF) on G using Talagrand's theory of majorizing measures [101]. Recall that the GFF on G with vertex set $V(G)$ is a centered Gaussian process $(\eta_x)_{x \in V(G)}$ with $\eta_{x_0} = 0$, for some $x_0 \in V(G)$, and covariance structure given by the Green kernel of the random walk killed at x_0.

Recent years have witnessed a growing interest in studying the geometric and analytic properties of random graphs partly motivated by applications in research areas ranging from sociology and systems biology to interacting particle systems as well as by the need to present convincing models to gain insight into real-world networks. One aspect of this development consists of examining the metric structure and connectivity of random graphs at criticality, that is precicely when we witness the emergence of a giant component that has size proportional to the number of vertices of the graph.

Several examples of trees and graphs, including critical Galton-Watson trees, possess Aldous' Brownian continuum random tree (CRT) as their scaling limit, see [6] and [77] (its universality class is, in fact, even larger, e.g. critical multi-type Galton-Watson trees [87], random trees with prescribed degree sequence satisfying certain conditions [27], random dissections [39], random graphs from subcritical classes [90]). Also, it appears as a building block of the limiting space of rescaled random quadrangulations, which is constructed as a complicated quotient of the Brownian CRT, see [80]. A program [22] has been launched having as its general aim to prove that the maximal components in the critical regime of a number of fundamental random graph models, with their distances scaling like $n^{1/3}$, fall into the basin of attraction of the Erdős-Rényi random graph. Their scaling limit is a multiple of the scaling limit of the Erdős-Rényi random graph in the critical window, which in turn is a tilted version of the Brownian CRT, where a finite number of vertices have been identified. Two of the examples that belong to the Erdős-Rényi universality class are the configuration model in the critical scaling window and critical inhomogeneous random graphs, where different vertices have different proclivity to form edges. We point out the recent work of [23] and [24] respectively.

In [37], Croydon, Hambly and Kumagai established criteria for the convergence of mixing times for random walks on general sequences of finite graphs. Furthermore, they applied their mixing time results in a number of examples of random graphs, such as self-similar fractal graphs with random weights, critical Galton-Watson trees, the critical Erdős-Rényi random graph and the range of high-dimensional random walk.

In Chapter 3, motivated by their approach, starting with the strong assumption that the sequences of graphs, associated measures, walks and local times converge appropriately, we provide asymptotic bounds on the distribution of the blanket times of the random walks in the sequence. The precise nature of these bounds ensures convergence of the ε-blanket times of the random walks if the ε-blanket time of the limiting diffusion is continuous with probability 1 at ε. To demonstrate our main results, in Chapter 4, this enables us to prove annealed convergence in various examples of critical random graphs, including critical Galton-Watson trees, the Erdős-Rényi random graph in the critical window and the configuration model in the scaling critical window. Our contribution refines the previous existing tightness results on the order of the blanket time (e.g. [5], [16]).

Another goal of the investigation is to provide a description for the scaling limits of stochastic processes on tree-like spaces, which in the last few years became well-understood. To lay out a distinctive but non-exhaustive list of particular cases, we cite some previous work on scaling limits of simple random walks on critical Galton-Watson trees, conditioned on their size, with finite [31] or infinite variance [33], the two-dimensional uniform spanning tree [15], and Λ-coalescent measure trees [13, Section 7.5]. Last but not least, in [73] diffusions on dendrites are constructed by approximating Dirichlet energies.

Despite the distinct characteristics of the processes mentioned, a shared feature is that their convergence essentially emanates from the convergence of metrics and measures that provide the natural scale functions and speed measures in this setting. Indeed, it was shown that the Gromov-Hausdorff-vague convergence (for a definition, see Section 2.3) of the metric measure trees and a certain non-explosion of the resistances [36], or a condition on the lengths of edges leaving compact sets [13] (neither condition implies the other, see [36, Remark 1.3(a)]) yields the convergence of the associated stochastic processes. For this very reason [13] and [36] can be seen as a generalization of Stone's invariance principle, who fifty years ago in [99] considered Markov processes which share the characteristic that their state spaces are closed subsets of the real line and that their random trajectories do not jump over points. Even more important, the result proved in [36] holds for other spaces (not necessarily tree-like) equipped with a resistance metric and a measure, allowing for a broader range of examples to be treated. Beyond the framework of resistance metrics, it parallels the recent work of Suzuki in [100] who showed that the pointed measured Gromov-Hausdorff convergence of a sequence of metric measure spaces that satisfy a Riemannian curvature-dimension condition, implies the weak convergence of the underlying Brownian motions. We would like to draw to the attention of the reader the complementary work of [38], where the stronger uniform volume growth (with volume doubling) condition enabled the study of time-changes of stochastic processes according to irregular measures, with the representative examples being the Liouville Brownian motion (in two dimensions, it is the diffusion associated with planar Liouville quantum gravity and is conjectured to be the scaling limit of simple random walks on random planar maps, see [21], [46] and [54]), the Bouchaud trap model, and the random conductance model model on a variety of self-similar trees and fractals. For the latter two models, the limiting process on the respective space is a FIN diffusion [52], which is connected with the localization and aging of physical spin systems, see [20] and [102].

Going a step further, it would be desirable to ask whether the distribution of a stochastic process that is not on natural scale is stable under perturbations in the geometry of the underlying spaces. To answer this question, it is possible to employ the framework of resistance metrics in order to study scaling limits of random walks in random environment on tree-like spaces. For a definition, see Section 5. The reversibility of this model offers an alternative description of it as an electrical network with conductances that can be described explicitly in terms of the potential of the random walk in random environment, see (5.2). This observation allows for random walks in random environment on tree-like spaces to be thought of as their associated variable speed random walks (the jump rate along edges is given by (5.8)) when the shortest path metric is replaced by a distorted metric, see (5.3), which is a resistance metric solely expressed in terms of the potential of the random walk in random environment, and endowed with an invariant measure specified in (5.4), which is a distortion of the uniform probability measure on the vertices of the tree.

In this case, Gromov-Hausdorff-vague convergence of the distorted metric measure trees, equipped with the potential of the random walk in random environment as a spatial element, can be viewed as a generalized metric measure version of Sinaĭ's regime in dimension one, that is when the potential converges to a two-sided Brownian motion. For a definition, see [104, Assumption 2.5.1]. Having this in mind, as an application of the main contributions in [13] and [36], the convergence of the distorted metrics and measures leads to the convergence of the the random walks in random environment. Here, we should stress that in the various examples we consider throughout the thesis, the limiting diffusion is a Brownian motion on a locally compact real tree, which is not on natural scale. Typically, keeping up with the terminology used to describe continuum analogues of one-dimensional random walks in random environment, it can be seen as a Brownian motion in random potential on a locally compact real tree.

In the one-dimensional model (for a definition, see Section 5.2), it is well-known that due to the large traps that arise, the random walk in random environment in Sinaĭ's regime localizes at a rate $(\log n)^2$ ((5.16) is due to [97], for sharp pathwise localization results, see [57]). Therefore, there is no hope in finding a Donsker's theorem in random environment without providing a discrete scheme that changes the random environment appropriately at every step. This was understood by Seignourel [95], who proved such a scheme for Sinaĭ's random walk, and verified a conjecture on the scaling limit of a random walk with infinitely many barriers dating back to Carmona [30]. Our approach is advantageous as it renders clear how the "flattening" of the environment that was introduced in the first place in [95], forces the potential to converge to a two-sided Brownian motion, and consequently the distorted metric and measure to converge to the scale function and the speed measure of the Brox diffusion [28], see (5.17). Also, we are able to considerably shorten Seignourel's proof but more importantly to remove the technical assumption of uniform ellipticity, see (5.15), and the assumption on the independent and identically distributed (i.i.d.) random environment as well.

Next, we consider (non-lattice) branching walk associated with a marked tree, that is a rooted ordered finite tree in which every edge is marked by a real value (it is equivalent to have values assigned to the vertices instead). We associate with each vertex a trajectory of a walk defined by summing the values of all the edges contained in the unique path from the root to that particular vertex (it is obvious that the walk is killed after as many steps as the height of the vertex evaluated at), see (5.33). The multiset of trajectories of the killed walk is called the branching walk. A branching random walk is constructed by choosing the skeleton and the values of the marked tree at random. We are interested in biased random walk on (non-lattice) branching random walk ϕ_n conditioned to have total population size n, where the underlying tree is a critical Galton-Watson tree T_n with exponential tails for the offspring distribution, and the values are independent, each distributed as a centered random variable Y, which has continuous distribution with fourth order polynomial tail decay. The bias, say $\beta > 1$, is chosen in such a way that the walk has a tendency to move towards a certain direction, see Section 5.4 and the details that lie therein. We prove that a weakly biased random walk on the aforementioned model converges to a Brownian motion in a random Gaussian potential on the CRT, which is a Brownian motion on the Brownian CRT endowed with a resistance metric, see (5.39) and a finite measure, see (5.40). For a definitive statement, see Theorem 5.4.2.

We believe that our work offers a promising candidate for the scaling limit of a biased random walk on the incipient infinite cluster (IIC) of Bernoulli-bond percolation on $\mathbb{Z}^d$ in high dimensions, that is when $d > 6$. At criticality, i.e. $p = p_c(d) \in (0,1)$, it is partially confirmed that there is no infinite open cluster. Instead, one could study random walks on the IIC:

$$\mathbb{P}_{\text{IIC}}(\cdot) = \lim_{n\to\infty} \mathbb{P}_{p_c}(\cdot | 0 \leftrightarrow \partial[-n,n]^d),$$

constructed in [68] for $d = 2$, and in [63] for $d \geq 11$, where $0 \leftrightarrow \partial[-n,n]^d$ translates to “there exists a (finite) path of open bonds connecting 0 and the boundary of the (ℓ_∞)-ball of radius n”. In high dimensions, the IIC is tree-like, its fractal dimension (with respect to the intrinsic metric) is 2, it has a unique backbone (the scaling limit of the backbone is identified as Brownian motion), and its scaling limit is related to super-processes or measure-valued diffusions, which are continuous-time and continuous-space processes that describe the random distribution of mass undergoing repartition and motion at the same time, see [58], [59] and [61]. Namely, in high dimensions, the scaling limit of the IIC is related to the integrated super-Brownian excursion (ISE) (defined by Aldous [7]). Take a critical branching random walk, condition on a large fixed total progeny where the generation structure of the population involved is ignored, and rescale space by $n^{-1/4}$. The scaling limit, which can be proven to exist is ISE, see (5.35).

Our declaration is justified in the sense that critical branching random walk is a mean-field model for percolation, and therefore it is expected that both models satisfy the same scaling properties. For an up-to-date survey, see [60]. The two are intuitively connected in the following way. In high dimensions, due to the vastness of the space, one could imagine that is relatively rare for a cycle to be discovered when exploring an open cluster vertex by vertex. Every vertex in percolation on the d-dimensional integer lattice has a number of neighbors distributed as a binomial with parameters $2d$ and p for which the edge leading to it is open. On the other hand, consider a branching random walk thought of as percolation on the $2d$-ary tree that is randomly embedded into $\mathbb{Z}^d$ by mapping the root of the $2d$-ary tree to the origin in $\mathbb{Z}^d$. Furthermore, an individual spatial location has increment chosen uniformly at random from the neighbors of the origin in $\mathbb{Z}^d$. Such a process only differs from percolation in $\mathbb{Z}^d$ in that it ignores cycles.

Attempting to give a plausible answer to [19, Question 5.3] posed by Ben Arous and Fribergh, the right scaling for a biased random walk on the IIC of $\mathbb{Z}^d$ is that of a random walk with a weak cartesian bias to a single direction, identical to the one introduced in Theorem 5.4.2, with the limit being a Brownian motion in a random Gaussian potential that maps an infinite version of the Brownian CRT to the Euclidean space, or alternatively, a Brownian motion in a random Gaussian potential on the ISE (the Brownian motion on the latter object was constructed for $d \geq 8$ by Croydon [32]). Just as critical branching random walk is a mean-field model for percolation, critical branching random walk conditioned on survival is a mean-field model for the high-dimensional IIC, which explains why an unbounded version of the Brownian CRT is expected to appear in the limit. As for establishing the corresponding limit for the weakly biased random walk on lattice (every edge is marked by an integer value) branching random walk, [18] outlines a program of four conditions to be checked in order to provide a flexible scaling theorem that will be generally applicable or adaptable to several models of large critical graphs. In this direction, it would be a meaningful project to check, as it was done for the simple random walk on the lattice branching random walk in [17], whether those conditions are satisfied, utilising the connection between distorted resistance metrics and random walks in random environment that the present thesis suggests.

Finally, we demonstrate an appealing application to non-Markovian settings. The edge-reinforced random walk (ERRW) was introduced by Coppersmith and Diaconis in 1986 (for references on the ERRW, see also [11], [43], [44], [67]) as a discrete process on the vertices of undirected graphs, starting from a fixed vertex. Given initial weights to all edges, whenever an edge is crossed the weight of that edge increases by one. The transition, through edges leading out of a particular vertex chosen, has probability proportional to their various (currently updated) weights. In the context of the ERRW on trees by Pemantle [91] (for a definition, see Section 5.5), due to the absence of cycles, the transitions of the process are decided by independent Pólya urns, one per vertex, where edges leading out play the role of colours and initial weights that of the number of balls of each colour. The ERRW on other undirected graphs by Sabot and Tarrès [93] is a random walk in a correlated, yet explicit, random environment.

It was not until recently that a scaling limit of the ERRW on the dyadic one-dimensional lattice appeared in [83]. The scaling limit introduced is a one-dimensional diffusion in a random potential that contains a scale-changed two-sided Brownian motion with a drift. We remark how their result can be recovered by using Theorem 5.2.3, which still holds when the limiting random potential in Assumption 6 has enough regularity for (5.24) and (5.25) to make sense. In addition, we introduce the scaling limit of the ERRW on a critical Galton-Watson tree T_n with finite variance, conditioned to have total population size n, as a Brownian motion in a random Gaussian potential with a drift given by the natural CRT-distance to the root. For a definitive statement see Theorem 5.5.4.

The large time behavior of the continuous space limit of the ERRW on $2^{-n}\mathbb{Z}$ was examined in [83]. Actually, the leading order is given by the deterministic drift part in the random potential, which is an artefact of the self-reinforcement and leaves the continuous space limit to oscillate between $-1/6$ and $1/6$ at a logarithmic rate.

In our model, the prospect to explore aging (a system ages when its decorrelation properties are age dependent: the older it gets the longer it takes to forget its past, in particular aging has been extensively studied in the context of spin-glass dynamics) and localization properties of the diffusions in random potential is meaningful only insofar as the Brownian CRT is replaced with its unbounded variant, Aldous' self-similar continuum random tree (SSCRT), which relates to the three-dimensional Bessel process BES(3) in the same way that the Brownian CRT relates to the normalized Brownian excursion, or to use Duquesne's terminology in [48], the SSCRT is a continuum random sin-tree coded by left and right height processes that are independent BES(3). This random tree is a size-biased random tree with Brownian branching mechanism that appears naturally as a continuous analogue of critical Galton-Watson trees conditioned on non-extinction (e.g. [70], [82]). As a result, to transfer our result in this setting, the scaling limit of the ERRW on critical Galton-Watson trees conditioned to survive (or "grow to infinity") is a Brownian motion in a random Gaussian potential with a drift given by the natural SSCRT-distance to the root, which as we discuss in Chapter 6, can be expected to localize (in probability) at the root at a logarithmic rate.

Chapter 2

Preliminaries

In this chapter, we introduce the necessary framework and several technical results that we use repeatedly throughout the thesis. We do not prove any new results in Section 2.1, although we give the definitions of metric measure trees, such as real trees coded by functions. In Section 2.2, we cover some known formulas of Itô's excursion measure of reflected Brownian motion and we survey a description of it that stresses its Markovian attributes. In Section 2.3, we define an extended Gromov-Hausdorff topology and derive some useful properties.

2.1 Real trees

The definitions of boundedly finite pointed metric measure trees appeared in the course of extending results that hold for real-valued Markov processes to Markov processes that take values in tree-like spaces. We refer to [13] for the preliminary work we do here.

A pointed metric space (T, r, ϱ) with a distinguished point ϱ is called Heine-Borel if (T, r) has the Heine-Borel property, i.e. each closed bounded set in T is compact. Note that this implies that (T, r) is complete, separable and locally compact.

Definition 2.1.1 (rooted metric measure trees). *A rooted metric tree is a pointed Heine-Borel space* (T, r, ϱ) *that satisfies the four point condition*

$$r(u_1, u_2) + r(u_3, u_4) \leq \max\{r(u_1, u_3) + r(u_2, u_4), r(u_1, u_4) + r(u_2, u_3)\},$$

for every $u_1, u_2, u_3, u_4 \in T$, and if for every $u_1, u_2, u_3 \in T$ there exists a unique point $u := u(u_1, u_2, u_3) \in T$, such that

$$r(u_i, u_j) = r(u_i, u) + r(u, u_j),$$

for every $i, j \in \{1, 2, 3\}$ with $i \neq j$. The point u is usually called the branch point, and the distinguished point ϱ is referred to as the root.

A rooted metric measure tree (T, r, ν, ϱ) is a rooted metric tree (T, r, ϱ) equipped with a measure ν that has full support on $(T, \mathcal{B}(T))$ and charges every bounded set with finite measure, if $\mathcal{B}(T)$ denotes the Borel σ-algebra of (T, r).

In a rooted metric tree (T, r, ϱ), for $x, y \in T$, we define the path intervals

$$[[x, y]] := \{z \in T : r(x, y) = r(x, z) + r(z, y)\},$$

$$[x, y]] := [[x, y]] \setminus \{x\}, \qquad [x, y] := [[x, y]] \setminus \{x, y\}.$$

If $x \neq y$ and $[[x, y]] = \{x, y\}$, we say that x and y are connected by an edge in T and use the notation $x \sim y$. Due to separability, a rooted metric tree can only have countably many edges. Denote the skeleton of (T, r, ϱ) as

$$\mathrm{Sk}(T) := \cup_{u \in T} [\varrho, u] \cup \mathrm{Is}(T),$$

where $\mathrm{Is}(T)$ is the set of isolated points of (T, r, ϱ), excluding the root. For any separable metric space that satisfies the four point condition, the notion of a length measure was introduced in [13]. In short, using that $\mathcal{B}(T)|_{\mathrm{Sk}(T)}$ is the smallest σ-algebra that contains all the open path intervals with endpoints in a countable dense subset of T, the validity of the following statement, which we turn into a definition, is justified.

Definition 2.1.2 (length measure)**.** *There exists a unique σ-finite measure λ on the rooted metric tree (T, r, ϱ), such that $\lambda(T \setminus \mathrm{Sk}(T)) = 0$ and for all $u \in T$,*

$$\lambda([\varrho, u]]) = r(\varrho, u).$$

Such a measure is called the length measure of (T, r, ϱ).

If (T, r) is a discrete tree, i.e. all the points in T are isolated, the length measure shifts the length of an edge to the endpoint that is further away from the root, and therefore it does depend on the root.

The first definitions of random real trees date back to Aldous [4]. Informally, real trees are metric trees without cycles that are locally isometric to the real line. We refer to [78] for a general presentation of the topic.

Definition 2.1.3 (real trees)**.** *A metric space (T, r) is a real tree if the two following properties hold for every $x, y \in T$.*

(i) It has a unique geodesic. There exists a unique isometry $f_{x,y} : [0, r(x, y)] \to T$ such that $f_{x,y}(0) = x$ and $f_{x,y}(r(x, y)) = y$.

(ii) It does not contain cycles. If $q : [0, 1] \to T$ is continuous and injective such that $q(0) = x$ and $q(1) = y$, then

$$q([0, 1]) = f_{x,y}([0, r(x, y)]).$$

A real tree has no edges. Therefore, if (T, r) is a real tree, then

$$\mathrm{Sk}(T) = \cup_{u,v \in T}[u, v]. \tag{2.1}$$

The unique length measure that extends the Lebesgue measure on the real line coincides with the trace onto $\mathrm{Sk}(T)$ of the one-dimensional Hausdorff measure on T. To describe a method to generate random real trees, which will play a crucial role to our forthcoming applications, we turn our attention first to a deterministic setting. Let $g : [0, \infty) \to [0, \infty)$ be a continuous function with compact support, such that $g(0) = 0$. We let

$$\mathrm{supp}(g) := \{t \geq 0 : g(t) > 0\},$$

denote the support of g. To avoid trivial cases, we assume that g is not identical to zero. For every $s, t \geq 0$, let $m_g(s, t) := \inf_{r \in [s \wedge t, s \vee t]} g(r)$ and $d_g : [0, \infty) \times [0, \infty) \to \mathbb{R}_+$ defined by

$$d_g(s, t) := g(s) + g(t) - 2m_g(s, t). \tag{2.2}$$

It is obvious that d_g is symmetric and satisfies the triangle inequality. One can introduce the equivalence relation $s \sim t$ if and only if $d_g(s, t) = 0$, or equivalently $g(s) = g(t) = m_g(s, t)$. Considering the quotient space

$$(\mathcal{T}_g, d_g) := ([0, \infty)_{/\sim}, d_g), \tag{2.3}$$

which we root at ϱ, the equivalence class of 0, it can be proven to be a rooted compact real tree, see [78, Theorem 2.1]. We use the term real tree coded by g to describe $\mathcal{T}_g$. If ζ is the supremum of supp(g), denote by $p_g : [0, \infty) \to \mathcal{T}_g$ the canonical projection, which is extended by setting $p_g(t) = \varrho$, for every $t \geq \zeta$. For every $A \in \mathcal{B}(\mathcal{T}_g)$, we let

$$\mu_{\mathcal{T}_g}(A) := \ell(\{t \geq 0 : p_g(t) \in A\}) \tag{2.4}$$

denote the image measure on $\mathcal{T}_g$ of the Lebesgue measure ℓ on $\mathbb{R}_+$ by the canonical projection p_g.

Definition 2.1.4 (spatial rooted metric measure trees)**.** *A d-dimensional spatial rooted metric measure tree is a pair $(\mathcal{T}, \phi)$, where $\mathcal{T} = (T, r, \nu, \varrho)$ is a rooted metric measure tree endowed with a continuous mapping $\phi : \mathcal{T} \to \mathbb{R}^d$.*

Note that the terminology spatial is borrowed from [49, Section 6].

2.2 Itô's excursion theory of Brownian motion

We recall some key facts of Itô's excursion theory of reflected Brownian motion collected in [79], [80] and [88].

A detailed account of the theory can be found in [92, Chapter XII]. Our main interest here lies on the scaling property of the Itô excursion measure. Let $(L_t^0)_{t\geq 0}$ denote the local time process at level 0 of the reflected Brownian motion $(|B_t|)_{t\geq 0}$, which can be defined by the approximation

$$L_t^0 = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{[0,\varepsilon]}(|B_s|) ds,$$

for every $t \geq 0$, a.s.

The local time process at level 0 is increasing, and its set of points of increase coincides with the set of time points for which the reflected Brownian motion is identical to zero. Now, introducing the right-continuous inverse of the local time process at level 0, i.e.

$$\tau_k := \inf\{t \geq 0 : L_t^0 > k\},$$

for every $k \geq 0$, we have that the set of points of increase of $(L_t^0)_{t\geq 0}$ alternatively belongs to the set

$$\{\tau_k : k \geq 0\} \cup \{\tau_{k-} : k \in D\},$$

where D is the set of countable discontinuities of the mapping $k \mapsto \tau_k$. For every $k \in D$ we define the excursion $(e_k(t))_{t\geq 0}$ with excursion interval (τ_{k-}, τ_k) away from 0 as

$$e_k(t) := \begin{cases} |B_{t+\tau_{k-}}| & \text{if } 0 \leq t \leq \tau_k - \tau_{k-}, \\ 0 & \text{if } t > \tau_k - \tau_{k-}. \end{cases}$$

Let E denote the space of excursions, namely the space of functions $e \in C(\mathbb{R}_+, \mathbb{R}_+)$, satisfying $e(0) = 0$ and $\zeta(e) := \sup\{s > 0 : e(s) > 0\} \in (0, \infty)$. By convention $\sup \emptyset = 0$. Observe that $e_k \in E$, and $\zeta(e_k) = \tau_k - \tau_{k-}$, for every $k \in D$.

The main theorem of Itô's excursion theory adapted in our setting is the existence of a σ-finite measure $\mathbb{N}(de)$ on the space of positive excursions of linear Brownian motion, such that the point measure

$$\sum_{k\in D} \delta_{(k,e_k)}(ds\ de)$$

is a Poisson measure on $\mathbb{R}_+ \times E$, with intensity $ds \otimes \mathbb{N}(de)$. The Itô excursion measure has the following scaling property. For every $a > 0$ consider the mapping $\Theta_a : E \to E$ defined by setting $\Theta_a(e)(t) := \sqrt{a}e(t/a)$, for every $e \in E$, and $t \geq 0$. Then,

$$\mathbb{N} \circ \Theta_a^{-1} = \sqrt{a}\mathbb{N}. \tag{2.5}$$

Versions of the Itô excursion measure $\mathbb{N}(de)$ under different conditionings are possible. For example one can define conditionings with respect to the height or the length of the excursion. For our purposes we focus on the fact that there exists a unique collection of probability measures $(\mathbb{N}_s : s > 0)$ on E, such that $\mathbb{N}_s(\zeta = s) = 1$, for every $s > 0$, and for every measurable event A of E,

$$\mathbb{N}(A) = \int_0^\infty \mathbb{N}_s(A) \frac{ds}{2\sqrt{2\pi s^3}}. \tag{2.6}$$

We might write $\mathbb{N}_1 = \mathbb{N}(\cdot|\zeta = 1)$ to denote the law of the normalized Brownian excursion. It is straightforward from (2.5) and (2.6) to check that $\mathbb{N}_s$ satisfies the scaling property

$$\mathbb{N}_s \circ \Theta_a^{-1} = \mathbb{N}_{as}. \tag{2.7}$$

To conclude our recap on Itô's excursion theory we highlight a description of $\mathbb{N}$ that emphasizes its Markovian properties. For $t > 0$ and $x, y \in \mathbb{R}$, let

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$$

be the Brownian transition density. For $t > 0$ and $x > 0$, let

$$q_t(x) = \partial_x p_t(x,y)|_{y=0} = \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}},$$

so that $t \mapsto q_t(x)$ is the density of the law of the first hitting time of x by B. For every integer $k \geq 1$, and every choice of $0 < t_1 < ... < t_k < 1$, and $x_1, ..., x_k > 0$, the distribution of $(e(t_1), ..., e(t_k))$ under $\mathbb{N}_1(de)$ has density

$$2\sqrt{2\pi} q_{t_1}(x_1) p^+_{t_2-t_1}(x_1, x_2) \cdots p^+_{t_k - t_{k-1}}(x_{k-1}, x_k) q_{1-t_k}(x_k),$$

where, for $t > 0$ and $x, y > 0$,

$$p_t^+(x,y) := p_t(x,y) - p_t(x,-y)$$

is the transition density of B killed at the first hitting time of 0.

2.3 Extended Gromov-Hausdorff topologies

In this section we define an extended Gromov-Hausdorff distance between quadruples consisting of a compact metric space, a Borel probability measure, a time-indexed right-continuous path with left-hand limits and a local time-type function. This allows us to make precise the assumption under which we are able to prove convergence of blanket times for the random walks on various models of critical random graphs. In Lemma 2.3.2, we give an equivalent characterization of Assumption 1 that will be used in Section 3.1 when proving distributional limits for the blanket times. Also, Lemma 2.3.3 will be useful when it comes to checking that the examples we treat satisfy Assumption 1.

Let (K, d_K) be a non-empty compact metric space. For a fixed $T > 0$, let X^K be a path in $D([0,T], K)$, the space of càdlàg functions, i.e. right-continuous functions with left-hand limits from $[0,T]$ to K.

Definition 2.3.1 (Skorohod metric)**.** *We say that a function* λ *from* $[0,T]$ *onto itself is a time-change if it is strictly increasing and continuous. Let* Λ *denote the set of all time-changes. If* $\lambda \in \Lambda$*, then* $\lambda(0) = 0$ *and* $\lambda(T) = T$*. We equip* $D([0,T],K)$ *with the Skorohod metric* d_{J_1} *defined as follows:*

$$d_{J_1}(x,y) := \inf_{\lambda\in\Lambda}\left\{\sup_{t\in[0,T]}|\lambda(t)-t| + \sup_{t\in[0,T]} d_K(x(\lambda(t)),y(t))\right\},$$

for $x, y \in D([0,T],K)$*.*

The idea behind going from the uniform metric to the Skorohod metric d_{J_1} is to say that two paths are close if they are uniformly close in $[0,T]$, after allowing small perturbations of time. Moreover, $D([0,T],K)$ endowed with d_{J_1} becomes a separable metric space, see [25, Theorem 12.2].

Definition 2.3.2 (standard Prokhorov metric)**.** *Let* $\mathcal{P}(K)$ *denote the space of Borel probability measures on* K*. If* $\mu,\nu \in \mathcal{P}(K)$*, we set*

$$d_P(\mu,\nu) = \inf\{\varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon, \text{ for } A \in \mathcal{M}(K)\},$$

where A^ε *is the* ε*-neighborhood of* A *and* $\mathcal{M}(K)$ *is the set of all closed subsets of* K*. This expression gives the standard Prokhorov metric between* μ *and* ν*.*

It is known that $(\mathcal{P}(K), d_P)$ is a Polish metric space, i.e. a complete and separable metric space, and the topology generated by d_P is exactly the topology of weak convergence, the convergence against bounded and continuous functionals, see [41, Appendix A.2.5]. Let π^K be a Borel probability measure on K and $L^K = (L_t^K(x))_{x\in K, t\in[0,T]}$ be a jointly continuous function of (t,x) taking positive real values. We say that two elements (K, π^K, X^K, L^K) and $(K', \pi^{K'}, X^{K'}, L^{K'})$ are equivalent, if there exists an isometry $f : K \to K'$ such that

- $\pi^K \circ f^{-1} = \pi^{K'}$,
- $f \circ X^K = X^{K'}$, which is a shorthand for $f(X_t^K) = X_t^{K'}$, for every $t \in [0,T]$.
- $L_t^{K'} \circ f = L_t^K$, for every $t \in [0,T]$, which is a shorthand for $L_t^{K'}(f(x)) = L_t^K(x)$, for every $t \in [0,T]$, $x \in K$.

Let $\mathbb{K}$ be the set of equivalence classes of quadruples (K, π^K, X^K, L^K) under the relation described above. We will often identify an equivalence class of $\mathbb{K}$ with a particular element of it.

Definition 2.3.3 (correspondence)**.** *A correspondence between K and K' is a subset of $K \times K'$, such that for every $x \in K$ there exists at least one x' in K' such that $(x, x') \in \mathcal{C}$, and conversely for every $x' \in K'$ there exists at least one $x \in K$ such that $(x, x') \in \mathcal{C}$.*

We now introduce a distance $d_{\mathbb{K}}$ on $\mathbb{K}$ by setting:

$$\begin{aligned} & d_{\mathbb{K}}((K, \pi^K, X^K, L^K), (K', \pi^{K'}, X^{K'}, L^{K'})) \\ & := \inf_{Z, \phi, \phi', \mathcal{C}} \Bigg\{ d_P^Z(\pi^K \circ \phi^{-1}, \pi^{K'} \circ \phi'^{-1}) + d_{J_1}^Z(\phi(X_t^K), \phi'(X_t^{K'})) \\ & + \sup_{(x,x') \in \mathcal{C}} \left(d_Z(\phi(x), \phi'(x')) + \sup_{t \in [0,T]} |L_t^K(x) - L_t^{K'}(x')| \right) \Bigg\}, \end{aligned}$$

where the infimum is taken over all metric spaces (Z, d_Z), isometric embeddings $\phi : K \to Z$, $\phi' : K' \to Z$ and correspondences $\mathcal{C}$ between K and K'. In the above expression d_P^Z is the standard Prokhorov distance between Borel probability measures on Z, and $d_{J_1}^Z$ is the Skorohod metric d_{J_1} between càdlàg paths on Z.

In the following proposition we check that the definition of $d_{\mathbb{K}}$ induces a metric and that the resulting metric space is separable. The latter fact will be used repeatedly later when it comes to applying Skorohod's representation theorem on sequences of random graphs to prove statements regarding their blanket times. Before proceeding to the proof of Proposition 2.3.1, let us first make a few remarks about the ideas behind the definition of $d_{\mathbb{K}}$. The first term along with the Hausdorff distance on Z between $\phi(K)$ and $\phi'(K')$ is that used in the Gromov-Hausdorff-Prokhorov distance for compact metric spaces, see [1, Section 2.2, (6)]. Though, in our definition of $d_{\mathbb{K}}$ we did not consider the Hausdorff distance between the embedded compact metric spaces K and K', it is absorbed by the first part of the third term in the expression for $d_{\mathbb{K}}$. Recall here the equivalent definition of the standard Gromov-Hausdorff distance via correspondences as a way to relate two compact metric spaces, see [29, Theorem 7.3.25]. The motivation for the second term comes from [34], where the author defined a distance between pairs of compact length spaces (for a definition of a length space see [29, Definition 2.1.6]) and continuous paths on those spaces. The restriction on length spaces is not necessary, as we will see later, apropos of the proof that $d_{\mathbb{K}}$ provides a metric. Considering càdlàg paths instead of continuous paths and replacing the uniform metric with the Skorohod metric d_{J_1} allows us to prove separability without assuming that (K, d_K) is a non-empty compact length space. The final term was first introduced in [49, Section 6] to define a distance between spatial trees equipped with a continuous function. For an approach that generalizes the Gromov-Hausdorff metric between metric spaces equipped with some additional structure, we recommend the recent work of Khezeli in [71] and [72].

Proposition 2.3.1. *$(\mathbb{K}, d_{\mathbb{K}})$ is a separable metric space.*

Proof. That $d_{\mathbb{K}}$ is non-negative and symmetric is obvious. To prove that is also finite, for any choice of (K, π^K, X^K, L^K), $(K', \pi^{K'}, X^{K'}, L^{K'})$ consider the disjoint union $Z = K \sqcup K'$ of K and K'. Then, set

$$d_Z(x, x') := \mathrm{diam}_K(K) + \mathrm{diam}_{K'}(K'),$$

for any $x \in K$, $x' \in K'$, where

$$\mathrm{diam}_K(K) := \sup_{y,z \in K} d_K(y, z)$$

denotes the diameter of K with respect to d_K. Since K and K' are compact, their diameters are finite. Therefore, d_Z is finite for any $x \in K$, $x' \in K'$. To conclude that $d_{\mathbb{K}}$ is finite, simply suppose that $\mathcal{C} = K \times K'$ is the trivial correspondence.

Next, we show that $d_{\mathbb{K}}$ is positive-definite. Let

$$(K, \pi^K, X^K, L^K) \text{ and } (K', \pi^{K'}, X^{K'}, L^{K'})$$

be in $\mathbb{K}$, such that

$$d_{\mathbb{K}}((K, \pi^K, X^K, L^K), (K', \pi^{K'}, X^{K'}, L^{K'})) = 0.$$

Then, for every $\varepsilon > 0$, there exist $Z, \phi, \phi', \mathcal{C}$ such that the sum of the quantities inside the infimum in the definition of $d_{\mathbb{K}}$ is bounded above by ε. Furthermore, there exists $\lambda_\varepsilon \in \Lambda$ such that the sum of the quantities inside the infimum in the definition of $d_{J_1}^Z$ is bounded above by 2ε. Recall that for every $t \in [0, T]$, $L_t^K : K \to \mathbb{R}_+$ is continuous and since K is compact, then it is also uniformly continuous. Therefore, there exists a $\delta \in (0, \varepsilon]$ such that

$$\sup_{\substack{x_1, x_2 \in K: \\ d_K(x_1, x_2) < \delta}} \sup_{t \in [0,T]} |L_t^K(x_1) - L_t^K(x_2)| \leq \varepsilon. \tag{2.8}$$

Now, let $(x_i)_{i\geq 1}$ be a dense sequence of disjoint elements in K. Since K is compact, there exists an integer N_ε such that the collection of open balls $(B_K(x_i,\delta))_{i=1}^{N_\varepsilon}$ covers K. Defining $A_1 = B_K(x_1,\delta)$ and $A_i = B_K(x_i,\delta) \setminus \cup_{j=1}^{i-1} B_K(x_j,\delta)$, for $i = 2,...,N_\varepsilon$, we have that $(A_i)_{i=1}^{N_\varepsilon}$ is a disjoint cover of K. Consider a function $f_\varepsilon : K \to K'$ by setting:

$$f_\varepsilon(x) := x'_i$$

on A_i, where x'_i is chosen such that $(x_i, x'_i) \in \mathcal{C}$, for $i = 1,...,N_\varepsilon$. Note that by definition f_ε is a measurable function defined on K. For any $x \in K$, such that $x \in A_i$ for some $i = 1,...,N_\varepsilon$, we have that

$$\begin{aligned} d_Z(\phi(x),\phi'(f_\varepsilon(x))) &= d_Z(\phi(x),\phi'(x'_i)) \\ &\leq d_Z(\phi(x),\phi(x_i)) + d_Z(\phi(x_i),\phi'(x'_i)) \leq \delta + \varepsilon \leq 2\varepsilon. \end{aligned} \tag{2.9}$$

From (2.9), it follows that for any $x \in K$ and $y \in K$,

$$\begin{aligned} |d_Z(\phi(x),\phi(y)) - d_Z(\phi'(f_\varepsilon(x)),\phi'(f_\varepsilon(y))| &\leq d_Z(\phi(y),\phi'(f_\varepsilon(y))) \\ &\quad + d_Z(\phi(x),\phi'(f_\varepsilon(x))) \leq 2\varepsilon + 2\varepsilon = 4\varepsilon. \end{aligned}$$

This immediately yields

$$\sup_{x,y\in K} |d_K(x,y) - d_{K'}(f_\varepsilon(x),f_\varepsilon(y))| \leq 4\varepsilon. \tag{2.10}$$

From (2.10), we deduce the bound

$$d_P^{K'}(\pi^K \circ f_\varepsilon^{-1},\pi^{K'}) \leq 5\varepsilon \tag{2.11}$$

for the Prokhorov distance between $\pi^K \circ f_\varepsilon^{-1}$ and $\pi^{K'}$ in K'. Using (2.8) and the fact that the last quantity inside the infimum in the definition of $d_{\mathbb{K}}$ is bounded above by ε, we deduce

$$\sup_{x\in K, t\in[0,T]} |L_t^K(x) - L_t^{K'}(f_\varepsilon(x))| \leq 2\varepsilon. \tag{2.12}$$

Using (2.9) and the fact that the second quantity in the infimum is bounded above by ε, we deduce that for any $t \in [0,T]$,

$$d_Z(\phi'(f_\varepsilon(X^K_{\lambda_\varepsilon(t)})), \phi'(X^{K'}_t)) \leq d_Z(\phi'(f_\varepsilon(X^K_{\lambda_\varepsilon(t)})), \phi(X^K_{\lambda_\varepsilon(t)})) \\ + d_Z(\phi(X^K_{\lambda_\varepsilon(t)}), \phi'(X^{K'}_t)) \leq 2\varepsilon + 2\varepsilon = 4\varepsilon.$$

Therefore,

$$\sup_{t\in[0,T]} d_{K'}(f_\varepsilon(X^K_{\lambda_\varepsilon(t)}), X^{K'}_t) \leq 4\varepsilon. \tag{2.13}$$

Using a diagonalization argument we can find a sequence $(\varepsilon_n)_{n\geq 1}$ such that $f_{\varepsilon_n}(x_i)$ converges to some limit $f(x_i) \in K'$, for every $i \geq 1$. From (2.10), we immediately get that $d_K(x_i, x_j) = d_{K'}(f(x_i), f(x_j))$, for every $i, j \geq 1$. By [29, Proposition 1.5.9], this map can be extended continuously to the whole of K. This shows that f is distance-preserving. Reversing the roles of K and K', we are able to find also a distance-preserving map from K' to K. Hence f is an isometry. We are now able to check that $\pi^K \circ f^{-1} = \pi^{K'}$, $L^{K'}_t \circ f = L^K_t$, for all $t \in [0,T]$, and $f \circ X^K = X^{K'}$. Since $f_{\varepsilon_n}(x_i)$ converges to $f(x_i)$ in K', we can find $\varepsilon' \in (0, \varepsilon]$ such that $d_{K'}(f_{\varepsilon'}(x_i), f(x_i)) \leq \varepsilon$, for $i = 1, ..., N_\varepsilon$. Recall that $(x_i)_{i=1}^{N_\varepsilon}$ is an ε-net in K. Then, for $i = 1, ..., N_\varepsilon$, such that $x \in A_i$, using (2.10) and the fact that f is an isometry, we deduce

$$d_{K'}(f_{\varepsilon'}(x), f(x)) \leq d_{K'}(f_{\varepsilon'}(x), f_{\varepsilon'}(x_i)) \\ + d_{K'}(f_{\varepsilon'}(x_i), f(x_i)) + d_{K'}(f(x_i), f(x)) \leq 7\varepsilon. \tag{2.14}$$

This, combined with (2.11) implies

$$d_P^{K'}(\pi^K \circ f^{-1}, \pi^{K'}) \leq d_P^{K'}(\pi^K \circ f^{-1}, \pi^K \circ f_{\varepsilon'}^{-1}) + d_P^{K'}(\pi^K \circ f_{\varepsilon'}^{-1}, \pi^{K'}) \leq 12\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\pi^K \circ f^{-1} = \pi^{K'}$. Moreover, from (2.12) and (2.14),

$$\begin{aligned} &\sup_{x\in K, t\in[0,T]} |L^K_t(x) - L^{K'}_t(f(x))| \\ &\leq \sup_{x\in K, t\in[0,T]} |L^K_t(x) - L^{K'}_t(f_{\varepsilon'}(x))| + \sup_{x\in K, t\in[0,T]} |L^{K'}_t(f_{\varepsilon'}(x)) - L^{K'}_t(f(x))| \\ &\leq 2\varepsilon + \sup_{\substack{x_1', x_2' \in K': \\ d_{K'}(x_1', x_2') \leq 7\varepsilon}} \sup_{t\in[0,T]} |L^{K'}_t(x_1') - L^{K'}_t(x_2')|. \end{aligned}$$

Now, this and the uniform continuity of $L^{K'}$ (replace L^K by $L^{K'}$ in (2.8)) gives

$L_t^{K'} \circ f = L_t^K$, for all $t \in [0,T]$. Finally, we verify that $f \circ X^K = X^{K'}$. For any $t \in [0,T]$,

$$\begin{aligned} d_{K'}(f(X^K_{\lambda_\varepsilon(t)}), X_t^{K'}) &\le d_{K'}(f(X^K_{\lambda_\varepsilon(t)}), f_{\varepsilon'}(X^K_{\lambda_\varepsilon(t)})) \\ &\quad + d_{K'}(f_{\varepsilon'}(X^K_{\lambda_\varepsilon(t)}), X_t^{K'}) \le 7\varepsilon + 4\varepsilon = 11\varepsilon, \end{aligned}$$

where we used (2.13) and (2.14). Therefore,

$$\sup_{t\in[0,T]} d_{K'}(f(X^K_{\lambda_\varepsilon(t)}), X_t^{K'}) \le 11\varepsilon. \tag{2.15}$$

Recall that $\sup_{t\in[0,T]} |\lambda_\varepsilon(t) - t| \le 2\varepsilon$. From this and (2.15), it follows that for every $t \in [0,T]$, there exists a sequence $(z_n)_{n\ge1}$, such that $z_n \to t$ and $d_{K'}(f(X^K_{z_n}), X_t^{K'}) \to 0$, as $n \to \infty$. If t is a continuity point of $f \circ X^K$, then $d_{K'}(f(X^K_{z_n}), f(X^K_t)) \to 0$, as $n \to \infty$. Thus, $f(X^K_t) = X_t^{K'}$. If $f \circ X^K$ has a jump at t and $(z_n)_{n\ge1}$ has a subsequence $(z_{n_k})_{k\ge1}$, such that $z_{n_k} \ge t$ for any $k \ge 1$, then $d_{K'}(f(X^K_{z_{n_k}}), X_t^{K'}) \to 0$, as $n \to \infty$, and $d_{K'}(f(X^K_{z_{n_k}}), f(X^K_t)) \to 0$, as $n \to \infty$. Therefore, $f(X^K_t) = X_t^{K'}$. Otherwise, $z_n < t$, for n large enough and $d_{K'}(f(X^K_{z_n}), f(X^K_{t-})) \to 0$, as $n \to \infty$, which implies $f(X^K_{t-}) = X_t^{K'}$. Essentially, what we have proved is that if $f \circ X^K$ has a jump, then either $f(X^K_t) = X_t^{K'}$ or $f(X^K_{t-}) = X_t^{K'}$. But, since $X^{K'}$ is càdlàg, $f \circ X^K = X^{K'}$. This completes the proof that the quadruples (K, π^K, X^K, L^K) and $(K', \pi^{K'}, X^{K'}, L^{K'})$ are equivalent in $(\mathbb{K}, d_{\mathbb{K}})$, and consequently that $d_{\mathbb{K}}$ is positive-definite.

For the triangle inequality we follow the proof of [29, Proposition 7.3.16], which proves the triangle inequality for the standard Gromov-Hausdorff distance. Let $\mathcal{K}^i = (K^i, \pi^i, X^i, L^i)$ be an element of $(\mathbb{K}, d_{\mathbb{K}})$ for $i = 1,2,3$. Suppose that

$$d_{\mathbb{K}}(\mathcal{K}^1, \mathcal{K}^2) < \delta_1.$$

Thus, there exists a metric space Z_1, isometric embeddings $\phi_{1,1} : K^1 \to Z_1$, $\phi_{2,1} : K^2 \to Z_1$ and a correspondence $\mathcal{C}_1$ between K^1 and K^2, such that the sum of the quantities inside the infimum that defines $d_{\mathbb{K}}$ is bounded above by δ_1. Similarly, if

$$d_{\mathbb{K}}(\mathcal{K}^2, \mathcal{K}^3) < \delta_2,$$

there exists a metric space Z_2, isometric embeddings $\phi_{2,2} : K^2 \to Z_2$, $\phi_{2,3} : K^3 \to Z_2$ and a correspondence $\mathcal{C}_2$ between K^2 and K^3, such that the sum of the quantities inside the infimum that defines $d_{\mathbb{K}}$ is bounded above by δ_2. Next, we set $Z = Z_1 \sqcup Z_2$ to be the disjoint union of Z_1 and Z_2 and we define a distance on Z in the following way. Let $d_{Z|Z_i \times Z_i} = d_{Z_i}$, for $i = 1, 2$, and for $x \in Z_1$, $y \in Z_2$ set

$$d_Z(x, y) := \inf_{z \in K^2} \{d_{Z_1}(x, \phi_{2,1}(z)) + d_{Z_2}(\phi_{2,2}(z), y)\}.$$

It is obvious that d_Z is symmetric and non-negative. It is also easy to check that d_Z satisfies the triangle inequality. Identifying points that are separated by zero distance and slightly abusing notation, we turn (Z, d_Z) into a metric space, which comes with isometric embeddings ϕ_i of Z_i for $i = 1, 2$. Using the triangle inequality of the Prokhorov metric on Z, gives us that

$$\begin{aligned}
&d_P^Z(\pi^1 \circ (\phi_1 \circ \phi_{1,1})^{-1}, \pi^3 \circ (\phi_2 \circ \phi_{3,2})^{-1}) \\
&\leq d_P^Z(\pi^1 \circ (\phi_1 \circ \phi_{1,1})^{-1}, \pi^2 \circ (\phi_1 \circ \phi_{2,1})^{-1}) \\
&+ d_P^Z(\pi^2 \circ (\phi_1 \circ \phi_{2,1})^{-1}, \pi^3 \circ (\phi_2 \circ \phi_{3,2})^{-1}).
\end{aligned}$$

Now, since $\phi_1(\phi_{2,1}(y)) = \phi_2(\phi_{2,2}(y))$, for all $y \in K^2$, we deduce

$$\begin{aligned}
&d_P^Z(\pi^1 \circ (\phi_1 \circ \phi_{1,1})^{-1}, \pi^3 \circ (\phi_2 \circ \phi_{3,2})^{-1}) \\
&\leq d_P^{Z_1}(\pi^1 \circ \phi_{1,1}^{-1}, \pi^2 \circ \phi_{2,1}^{-1}) + d_P^{Z_2}(\pi^2 \circ \phi_{2,2}^{-1}, \pi^3 \circ \phi_{3,2}^{-1}).
\end{aligned} \tag{2.16}$$

A similar bound also applies to the embedded càdlàg paths. Namely, using the same methods as above, we deduce

$$\begin{aligned}
&d_{J_1}^Z((\phi_1 \circ \phi_{1,1})(X^1), (\phi_2 \circ \phi_{3,2})(X^3)) \\
&\leq d_{J_1}^Z(\phi_{1,1}(X^1), \phi_{2,1}(X^2)) + d_{J_1}^Z(\phi_{2,2}(X^2), \phi_{3,2}(X^3)).
\end{aligned} \tag{2.17}$$

Now, let

$$\mathcal{C} := \{(x, z) \in K^1 \times K^3 : (x, y) \in \mathcal{C}_1, (y, z) \in \mathcal{C}_2, \text{ for some } y \in K^2\}.$$

Observe that $\mathcal{C}$ is a correspondence between K^1 and K^3. Then, if $(x, z) \in \mathcal{C}$, there exists $y \in K^2$ such that $(x, y) \in \mathcal{C}_1$ and $(y, z) \in \mathcal{C}_2$, and noting again that $\phi_1(\phi_{2,1}(y)) = \phi_2(\phi_{2,2}(y))$, for all $y \in K^2$, we deduce

$$d_Z(\phi_1(\phi_{1,1}(x)), \phi_2(\phi_{3,2}(z))) \leq d_{Z_1}(\phi_{1,1}(x), \phi_{2,1}(y)) + d_{Z_2}(\phi_{2,2}(y), \phi_{3,2}(z)). \tag{2.18}$$

Using the same arguments one can prove a corresponding bound involving L^i, $i = 1,2,3$. Namely, if $(x,z) \in \mathcal{C}$, there exists $y \in K^2$ such that $(x,y) \in \mathcal{C}_1$ and $(y,z) \in \mathcal{C}_2$, and moreover

$$\sup_{t\in[0,T]} |L_t^1(x) - L_t^3(z)| \leq \sup_{t\in[0,T]} |L_t^1(x) - L_t^2(y)| + \sup_{t\in[0,T]} |L_t^2(y) - L_t^3(z)|. \tag{2.19}$$

Putting (2.16), (2.17), (2.18) and (2.19) together gives

$$d_{\mathbb{K}}(\mathcal{K}^1, \mathcal{K}^3) \leq \delta_1 + \delta_2,$$

and the triangle inequality follows. Thus, $(\mathbb{K}, d_{\mathbb{K}})$ forms a metric space.

To finish the proof, we need to show that $(\mathbb{K}, d_{\mathbb{K}})$ is separable. Consider an element (K, π, X, L) of $\mathbb{K}$. First, let K^n be a finite n^{-1}-net of K, which exists since K is compact. Furthermore, we can endow K^n with a metric d_{K^n}, such that $d_{K^n}(x,y) \in \mathbb{Q}$, and moreover $|d_{K^n}(x,y) - d_K(x,y)| \leq n^{-1}$, for every $x, y \in K^n$. We can choose a partition for K, $(A_x)_{x\in K^n}$, such that $x \in A_x$, and $\mathrm{diam}_K(A_x) \leq 2n^{-1}$. We can even choose the partition in such a way that A_x is measurable for all $x \in K^n$ (see for example the definition of $(A_i)_{i=1}^{N_\varepsilon}$ after (2.8)). Next, we construct a Borel probability measure π^n in K^n that takes rational mass at each point, i.e. $\pi^n(\{x\}) \in \mathbb{Q}$, and $|\pi^n(\{x\}) - \pi(A_x)| \leq n^{-1}$. Define ε_n by

$$\varepsilon_n := \sup_{\substack{s,t\in[0,T]: \\ |s-t|\leq n^{-1}}} \sup_{\substack{x,x'\in K: \\ d_K(x,x')\leq n^{-1}}} |L_s(x) - L_t(x')|.$$

By the joint continuity of L, $\varepsilon_n \to 0$, as $n \to \infty$. Let $0 = s_0 < s_1 < \cdots < s_r = T$ be a set of rational times such that $|s_{i+1} - s_i| \leq n^{-1}$, for $i = 0, ..., r-1$. Choose $L_{s_i}^n(x) \in \mathbb{Q}$ with $|L_{s_i}^n(x) - L_{s_i}(x)| \leq n^{-1}$, for every $x \in K^n$. We interpolate linearly between the finite collection of rational time points in order to define L^n to the whole domain $K^n \times [0,T]$. Let $\mathcal{C}^n := \{(x,x') \in K \times K^n : d_K(x,x') \leq n^{-1}\}$. Clearly $\mathcal{C}^n$ defines a correspondence between K and K^n. Let $(x,x') \in \mathcal{C}^n$ and $s \in [s_i, s_{i+1}]$, for some $i = 0, ..., r-1$. Then, using the triangle inequality we observe that

$$\begin{aligned} |L_s^n(x) - L_s(x')| &\leq |L_s^n(x) - L_s(x)| + |L_s(x) - L_s(x')| \\ &\leq |L_s^n(x) - L_s(x)| + \varepsilon_n. \end{aligned} \tag{2.20}$$

Since we interpolated linearly to define L^n beyond rational time points on the whole space $K^n \times [0,T]$, we have that

$$|L^n_s(x) - L_s(x)| \leq |L^n_{s_{i+1}}(x) - L_s(x)| + |L^n_{s_i}(x) - L_s(x)|. \tag{2.21}$$

Applying the triangle inequality again yields

$$\begin{aligned}|L^n_{s_i}(x) - L_s(x)| &\leq |L^n_{s_i}(x) - L_{s_i}(x)| + |L_{s_i}(x) - L_s(x)| \\ &\leq n^{-1} + \varepsilon_n.\end{aligned}$$

The same upper bound applies for $|L^n_{s_{i+1}}(x) - L_s(x)|$, and from (2.20) and (2.21) we conclude that for $(x,x') \in \mathcal{C}^n$ and $s \in [s_i, s_{i+1}]$, for some $i = 0, ..., r-1$,

$$|L^n_s(x) - L_s(x')| \leq 2n^{-1} + 3\varepsilon_n.$$

For $X \in D([0,T],K)$ and $A \subseteq [0,T]$ put

$$w(X;A) := \sup_{s,t\in A} d_K(X_t, X_s).$$

Now, for $\delta \in (0,1)$, define the càdlàg modulus to be

$$w'(X;\delta) := \inf_{\Sigma} \max_{1\leq i\leq k} w(X;[t_{i-1},t_i)),$$

where the infimum is taken over all partitions $\Sigma = \{0 = t_0 < t_1 < \cdots < t_k = T\}$, $k \in \mathbb{N}$, with $\min_{1\leq i\leq k}(t_i - t_{i-1}) > \delta$. For a function to lie in $D([0,T],K)$, it is necessary and sufficient to satisfy $w'(X;\delta) \to 0$, as $\delta \to 0$, see [25, Lemma 1, p.122-123]. Let B_n be the set of functions having a constant value in K^n over each interval $[(u-1)T/n, uT/n)$, for some $n \in \mathbb{N}$ and also a value in K^n at time T. Take $B = \cup_{n\geq 1} B_n$, and observe that is countable. Clearly, putting $z = (z_u)_{u=0}^n$, with $z_u = uT/n$, for every $u = 0, ..., n$ satisfies $0 = z_0 < z_1 < \cdots < z_n = T$. Let $T_z : D([0,T],K) \to D([0,T],K)$ be the map that is defined in the following way. For $X \in D([0,T],K)$ take $T_z X$ to have a constant value $X(z_{u-1})$ over the interval $[z_{u-1}, z_u)$ for $1 \leq u \leq n$ and the value $X(T)$ at $t = T$. From an adaptation of [25, Lemma 3, p.127], considering càdlàg paths that take values on metric spaces, we have that

$$d_{J_1}(T_z X, X) \leq Tn^{-1} + w'(X;Tn^{-1}). \tag{2.22}$$

Also, there exists $X^n \in B_n$, for which

$$d_{J_1}(T_z X, X^n) \leq T n^{-1}. \tag{2.23}$$

Combining (2.22) and (2.23), we have that

$$d_{J_1}(X^n, X) \leq d_{J_1}(X^n, T_z X) + d_{J_1}(T_z X, X) \leq 2Tn^{-1} + w'(X; Tn^{-1}).$$

With the choice of the sequence (K^n, π^n, X^n, L^n), we find that

$$d_{\mathbb{K}}((K^n, \pi^n, X^n, L^n), (K, \pi, X, L)) \leq (4 + 2T)n^{-1} + 3\varepsilon_n + w'(X; Tn^{-1}).$$

Recalling that $w'(X; Tn^{-1}) \to 0$, as $n \to \infty$, and noting that our sequence was drawn from a countable subset of $\mathbb{K}$ completes the proof of the proposition.

$\square$

Fix $T > 0$. Let $\tilde{\mathbb{K}}$ be the space of quadruples of the form (K, π^K, X^K, L^K), where K is a non-empty compact pointed metric space with a distinguished vertex ϱ, π^K is a Borel probability measure on K, $X^K = (X_t^K)_{t \in [0,K]}$ is a càdlàg path on K and $L^K = (L_t(x))_{x \in K, t \in [0,T]}$ is a jointly continuous positive real-valued function of (t, x). We say that two elements of $\tilde{\mathbb{K}}$, say (K, π^K, X^K, L^K) and $(K', \pi^{K'}, X^{K'}, L^{K'})$, are equivalent if and only there is a root-preserving isometry $f : K \to K'$, such that $f(\varrho) = \varrho'$, $\pi^K \circ f^{-1} = \pi^{K'}$, $f \circ X^K = X^{K'}$ and $L_t^{K'} \circ f = L_t^K$, for every $t \in [0, T]$. It is possible to define a metric on the equivalence classes of $\tilde{\mathbb{K}}$ by imposing in the definition of $d_{\mathbb{K}}$ that the infimum is taken over all correspondences that contain (ϱ, ϱ'). The incorporation of distinguished points to the extended Gromov-Hausdorff topology leaves the proof of Proposition 2.3.1 unchanged and it is possible to show that $(\tilde{\mathbb{K}}, d_{\tilde{\mathbb{K}}})$ is a separable metric space.

The aim of the following lemmas is to establish a sufficient condition for Assumption 1 to hold, as well as to show that if Assumption 1 holds then we can isometrically embed the rescaled graphs, measures, random walks and local times into a common metric space such that they all converge to the relevant objects. To be more precise we formulate this last statement in the next lemma.

Lemma 2.3.2. *If Assumption 1 is satisfied, then we can find isometric embeddings* *of $(V(G^n), d_{G^n})_{n\geq 1}$ and (K, d_K) into a common metric space (F, d_F) such that*

$$\lim_{n\to\infty} d_H^F(V(G^n), K) = 0, \qquad \lim_{n\to\infty} d_F(\varrho^n, \varrho) = 0, \tag{2.24}$$

where d_H^F is the standard Hausdorff distance between $V(G^n)$ and K, regarded as subsets of (F, d_F),

$$\lim_{n\to\infty} d_P^F(\pi^n, \pi) = 0, \tag{2.25}$$

where d_P^F is the standard Prokhorov distance between $V(G^n)$ and K, regarded as subsets of (F, d_F),

$$\lim_{n\to\infty} d_{J_1}^F(X^n, X) = 0, \tag{2.26}$$

where $d_{J_1}^F$ is the Skorohod d_{J_1} metric between càdlàg functions on $V(G^n)$ and K, regarded as subsets of (F, d_F). Also,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{\substack{x^n\in V(G^n), x\in K:\\ d_F(x^n,x)<\delta}} \sup_{t\in[0,T]} |L^n_{\beta(n)t}(x^n) - L_t(x)| = 0. \tag{2.27}$$

Proof. Since Assumption 1 holds, for each $n \geq 1$ we can find metric spaces (F_n, d_n), isometric embeddings $\phi_n : V(G^n) \to F_n$, $\phi'_n : K \to F_n$ and correspondences $\mathcal{C}^n$ (that contain (ϱ^n, ϱ)) between $V(G^n)$ and K such that (identifying the relevant objects with their embeddings)

$$d_P^{F_n}(\pi^n, \pi) + d_{J_1}^{F_n}(X^n, X) + \sup_{(x,x')\in\mathcal{C}^n} \left(d_n(x, x') + \sup_{t\in[0,T]} |L^n_{\beta(n)t}(x) - L_t(x')| \right) \leq \varepsilon_n, \tag{2.28}$$

where $\varepsilon_n \to 0$, as $n \to \infty$. Now, let $F = \sqcup_{n\geq 1} F_n$ be the disjoint union of F_n, and define the distance $d_{F|F_n\times F_n} = d_n$, for $n \geq 1$, and for $x \in F_n$, $x' \in F_{n'}$, $n \neq n'$

$$d_F(x, x') := \inf_{y\in K}\{d_n(x, y) + d_{n'}(y, x')\}.$$

This distance, as the distance that was defined in order to prove the triangle inequality in Proposition 2.3.1, is symmetric and non-negative, so identifying points that are separated by a zero distance, we turn (F, d_F) into a metric space, which comes with natural isometric embeddings of $(V(G^n), d_{G^n})_{n\geq 1}$ and (K, d_K). In this setting, under the appropriate isometric embeddings (2.24), (2.25) and (2.26) readily hold from (2.28). Thus, it only remains to prove (2.27). For every $x \in V(G^n)$, since $\mathcal{C}^n$ is a correspondence in $V(G^n) \times K$, there exists an $x' \in K$ such that $(x, x') \in \mathcal{C}^n$. Then, (2.28) implies that $d_F(x, x') \leq \varepsilon_n$. Now, let $(y, y') \in \mathcal{C}^n$, $(z, z') \in \mathcal{C}^n$ and note that

$$\begin{aligned}
&\sup_{t\in[0,T]} |L^n_{\beta(n)t}(y) - L^n_{\beta(n)t}(z)| \\
&\leq \sup_{t\in[0,T]} |L^n_{\beta(n)t}(y) - L_t(y')| + \sup_{t\in[0,T]} |L^n_{\beta(n)t}(z) - L_t(z')| + \sup_{t\in[0,T]} |L_t(y') - L_t(z')| \\
&\leq 2\varepsilon_n + \sup_{t\in[0,T]} |L_t(y') - L_t(z')|.
\end{aligned}$$

For any $\delta > 0$ and $y, z \in V(G^n)$, such that $d_{G^n}(y, z) < \delta$, we have that

$$d_K(y', z') \leq d_F(y, y') + d_F(z, z') + d_{G^n}(y, z) < 2\varepsilon_n + \delta.$$

Therefore,

$$\begin{aligned}
&\sup_{\substack{y,z\in V(G^n):\\ d_{G^n}(y,z)<\delta}} \sup_{t\in[0,T]} |L^n_{\beta(n)t}(y) - L^n_{\beta(n)t}(z)| \\
&\leq 2\varepsilon_n + \sup_{\substack{y,z\in K:\\ d_K(y,z)<2\varepsilon_n+\delta}} \sup_{t\in[0,T]} |L_t(y) - L_t(z)|.
\end{aligned} \tag{2.29}$$

Also, for every $x \in K$ there exists an $x' \in V(G^n)$ such that $(x', x) \in \mathcal{C}^n$, and furthermore $d_F(x', x) \leq \varepsilon_n$. Let $x^n \in V(G^n)$ such that $d_F(x^n, x) < \delta$. Then,

$$d_F(x^n, x') \leq d_F(x^n, x) + d_F(x', x) < 2\epsilon_n + \delta.$$

More generally, we have the following inclusion:

$$B_F(x, \delta) \cap V(G^n) \subseteq B_F(x', 2\varepsilon_n + \delta) \cap V(G^n).$$

For $x \in K$, and $x' \in V(G^n)$ with $d_F(x', x) \leq \varepsilon_n$, using (2.28), we deduce

$$\begin{aligned}
&\sup_{t\in[0,T]} |L^n_{\beta(n)t}(x^n) - L_t(x)| \\
&\leq \sup_{t\in[0,T]} |L^n_{\beta(n)t}(x^n) - L^n_{\beta(n)t}(x')| + \sup_{t\in[0,T]} |L^n_{\beta(n)t}(x') - L_t(x)| \\
&\leq \varepsilon_n + \sup_{t\in[0,T]} |L^n_{\beta(n)t}(x^n) - L^n_{\beta(n)t}(x')|.
\end{aligned}$$

Since $x^n \in B_F(x', 2\varepsilon_n + \delta) \cap V(G^n)$, taking the supremum over all $x^n \in V(G^n)$ and $x \in K$, for which $d_F(x^n, x) < \delta$ and using (2.29), we deduce

$$\begin{aligned}
&\sup_{\substack{x^n \in V(G^n), x \in K: \\ d_F(x^n, x) < \delta}} \sup_{t \in [0,T]} |L^n_{\beta(n)t}(x^n) - L_t(x)| \\
&\leq \varepsilon_n + \sup_{\substack{y,z \in V(G^n): \\ d_{G^n}(y,z) < 2\varepsilon_n + \delta}} \sup_{t \in [0,T]} |L^n_{\beta(n)t}(y) - L^n_{\beta(n)t}(z)| \\
&\leq 3\varepsilon_n + \sup_{\substack{y,z \in K: \\ d_K(y,z) < 4\varepsilon_n + \delta}} \sup_{t \in [0,T]} |L_t(y) - L_t(z)|.
\end{aligned}$$

Using the continuity of L, as $n \to \infty$

$$\limsup_{n \to \infty} \sup_{\substack{x^n \in V(G^n), x \in K: \\ d_F(x^n, x) < \delta}} \sup_{t \in [0,T]} |L^n_{\beta(n)t}(x^n) - L_t(x)| \leq \sup_{\substack{y,z \in K: \\ d_K(y,z) \leq \delta}} \sup_{t \in [0,T]} |L_t(y) - L_t(z)|. \tag{2.30}$$

Again appealing to the continuity of L, the right-hand side converges to 0, as $\delta \to 0$. Thus, we showed that (2.27) holds, and this finishes the proof of Lemma 2.3.2.

□

In the process of proving (2.27) we established a useful equicontinuity property. We state and prove this property in the next corollary.

Corollary 2.3.2.1. *Fix $T > 0$ and suppose that Assumption 1 holds. Then,*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \sup_{\substack{y,z \in V(G^n): \\ d_{G^n}(y,z) < \delta}} \sup_{t \in [0,T]} |L^n_{\beta(n)t}(y) - L^n_{\beta(n)t}(z)| = 0. \tag{2.31}$$

Proof. As we hinted upon when deriving (2.30), using the continuity of L,

$$\limsup_{n \to \infty} \sup_{\substack{y,z \in V(G^n): \\ d_{G^n}(y,z) < \delta}} \sup_{t \in [0,T]} |L^n_{\beta(n)t}(y) - L^n_{\beta(n)t}(z)| \leq \sup_{\substack{y,z \in K: \\ d_K(y,z) \leq \delta}} \sup_{t \in [0,T]} |L_t(y) - L_t(z)|.$$

Sending $\delta \to 0$ gives the desired result.

□

Next, we prove that if we reverse the conclusions of Lemma 2.3.2, more specifically if (2.24)-(2.27) hold, then also Assumption 1 holds.

Lemma 2.3.3. *Suppose that* (2.24)-(2.27) *hold. Then so does Assumption 1.*

Proof. There exist isometric embeddings of $(V(G^n), d_{G^n})_{n\geq 1}$ and (K, d_K) into a common metric space (F, d_F), under which the assumptions (2.24)-(2.27) hold. Since (2.24) gives the convergence of spaces under the Hausdorff metric, (2.25) gives the convergence of measures under the Prokhorov metric and (2.26) gives the convergence of paths under d_{J_1}, it only remains to check the uniform convergence of local times. Let $\mathcal{C}^n$ be the set of all pairs $(x, x') \in K \times V(G^n)$, for which $d_F(x, x') \leq n^{-1}$. Since (2.24) holds, $\mathcal{C}^n$ are correspondences for $n \geq 1$. Then, for $(x, x') \in \mathcal{C}^n$

$$\sup_{t\in[0,T]} |L^n_{\beta(n)t}(x') - L_t(x)| \leq \sup_{\substack{x^n \in V(G^n), x\in K: \\ d_F(x^n,x)<n^{-1}}} \sup_{t\in[0,T]} |L^n_{\beta(n)t}(x^n) - L_t(x)|,$$

and using (2.27) completes the proof.

□

2.3.1 Topological considerations

For two fixed metric spaces (K, d_K) and $(K', d_{K'})$ and a subset $\mathcal{C} \subseteq K \times K'$, the distortion of $\mathcal{C}$ is defined as

$$\mathrm{dis}(\mathcal{C}) := \sup\{|d_K(x, y) - d_{K'}(x', y')| : (x, x'), (y, y') \in \mathcal{C}\}.$$

Given a Borel probability measure π on $K \times K'$, with marginals π_1 and π_2, the discrepancy of π with respect to π^K and $\pi^{K'}$ is defined as

$$D(\pi; \pi^K, \pi^{K'}) := ||\pi_1 - \pi^K||_{\mathrm{TV}} + ||\pi_2 - \pi^{K'}||_{\mathrm{TV}},$$

where $||\cdot||_{\mathrm{TV}}$ denotes the total variation distance between signed measures. If π^K and $\pi^{K'}$ are probability distributions, a Borel probability measure π on $K \times K'$ is a coupling of π^K and $\pi^{K'}$ in the standard sense, if $D(\pi; \pi^K, \pi^{K'}) = 0$. The following lemma gives an alternative description of $d_{\tilde{\mathbb{K}}}$.

Lemma 2.3.4. *The metric* $d_{\tilde{\mathbb{K}}}$ *between* (K, π^K, L^K) *and* $(K', \pi^{K'}, L^{K'})$ *is also* *given by:*

$$d_{\tilde{\mathbb{K}}}((K,\pi^K,L^K),(K',\pi^{K'},L^{K'}))$$
$$:= \inf_{\substack{\pi,\mathcal{C}:\\(\varrho,\varrho')\in\mathcal{C}}} \left\{\frac{1}{2}\mathrm{dis}(\mathcal{C}) + D(\pi;\pi^K,\pi^{K'}) + \pi(\mathcal{C}^c) + \sup_{(z,z')\in\mathcal{C}} ||L^K(z) - L^{K'}(z')||_{\infty,[0,T]}\right\},$$

where the infimum is taken over all correspondences and Borel probability measures on $K \times K'$.

Given a metric space (Z, d_Z) and isometric embeddings $\phi : K \to Z$, $\phi' : K' \to Z$, recall that the standard Prokhorov distance between $\pi^K \circ \phi^{-1}$ and $\pi^{K'} \circ \phi'^{-1}$ on the common metric space (Z, d_Z) appeared in the definition of $d_{\tilde{\mathbb{K}}}$. Another distance, which fits to the setting where π^K and $\pi^{K'}$ are not supported in the same metric space, but still generates the same topology, is given by

$$\inf\big\{\varepsilon > 0 : D(\pi;\pi^K\circ\phi^{-1},\pi^{K'}\circ\phi'^{-1}) < \varepsilon,$$
$$\pi(\{(z,z') : d_Z(z,z') \geq \varepsilon\}) < \varepsilon, \text{ for a probability measure } \pi \text{ on } Z\big\}.$$

To extend this, the condition $\pi(\{(z,z') : d_Z(z,z') \geq \varepsilon\}) < \varepsilon$ is replaced by $\pi(\mathcal{C}^c) < \varepsilon$, an analogous condition on the set of pairs lying outside the correspondence $\mathcal{C}$, measured by π.

Remark. *In Lemma 2.3.4, if the infimum is taken over all correspondences between K and K', and couplings π on $K \times K'$, observe that the formulation of $d_{\tilde{\mathbb{K}}}$ is simplified not to include $D(\pi;\pi^K,\pi^{K'})$.*

To extend $d_{\tilde{\mathbb{K}}}$ to a metric between (non-compact) Heine-Borel metric spaces consider restrictions of $((K, d_K, \varrho), \pi^K, L^K)$ to $\bar{B}_K(\varrho, R) := \{u \in K : d_K(\varrho, u) \leq R\}$, the closed ball of radius R centred at the root ϱ, denoted by

$$(\mathcal{K}, L^K)|_R := \left(\left(\bar{B}_K(\varrho,R), d_K|_{\bar{B}_K(\varrho,R)\times\bar{B}_K(\varrho,R)}, \varrho\right), \pi^K(\cdot\cap\bar{B}_K(\varrho,R)), L^K|_{\bar{B}_K(\varrho,R)}\right).$$

By assumption $\left(\bar{B}_K(\varrho,R), d_K|_{\bar{B}_K(\varrho,R)\times\bar{B}_K(\varrho,R)}, \varrho\right)$ is compact, and thus $(\mathcal{K}, L^K)|_R \in \tilde{\mathbb{K}}$. The function defined by setting:

$$\int_0^\infty e^{-R}\left(d_{\tilde{\mathbb{K}}}\left((\mathcal{K},L^K)|_R,(\mathcal{K}',L^{K'})|_R\right)\wedge 1\right)dR$$

is well-defined, see [1, Lemma 2.8]. Moreover, it can be checked that it is a metric. For each $n \in \mathbb{N} \cup \{\infty\}$, let $(\mathcal{K}^n, L^n) := ((K^n, d_{K^n}, \pi^n, \varrho^n), L^n)$. We say that $(\mathcal{K}^n, L^n)$ converges to $(\mathcal{K}^\infty, L^\infty)$ in the spatial Gromov-Hausforff-vague topology if and only if, for Lebesgue-almost-every $R \geq 0$,

$$d_{\tilde{\mathbb{K}}}((\mathcal{K}^n, L^n)|_R, (\mathcal{K}^\infty, L^\infty)|_R) \to 0.$$

In a number of settings, for instance, in studying the weakly biased random walk on the range of critical branching random walk in Section 5.4, it is relevant to consider the embedding into Euclidean space. Also, many self-similar fractals are naturally defined as subsets of $\mathbb{R}^d$ or some other metric space, and it might sometimes be more desirable to state the convergence of graphs to such fractals in that space, instead of an abstract metric space isometric to their associated metrics. To take this on account, one can adapt the Gromov-Hausdorff-vague topology to include the case in which the spaces of interest are embedded into a common metric space when the relevant embeddings are continuous (but not necessarily isometric) with respect to the metric that the spaces are endowed with. To incorporate collections of spatial rooted metric measure spaces of the form $((T, r, \nu, \varrho), \varphi)$ to the Gromov-Hausdorff-vague topology, where $\varphi : T \to (K, d_K)$ is a given continuous embedding of T into a complete, separable metric space (K, d_K), is equivalent to viewing $((T, r, \nu, \varrho), \varphi)$ as belonging to $\tilde{\mathbb{K}}$, where the spatial element now consists of φ, rather than a local time-type function. More specifically the metric $d_{\tilde{\mathbb{K}}}$ is modified to measure the distance between such $((T, r, \nu, \varrho), \varphi)$ and $((T', r', \nu', \varrho'), \varphi')$ by

$$\inf_{\substack{\pi, \mathcal{C}: \\ (\varrho, \varrho') \in \mathcal{C}}} \left\{ \frac{1}{2} \mathrm{dis}(\mathcal{C}) + D(\pi; \nu, \nu') + \pi(\mathcal{C}^c) + \sup_{(z, z') \in \mathcal{C}} d_K(\varphi(z), \varphi'(z')) \right\},$$

where the infimum is taken over all correspondences and Borel probability measures on $T \times T'$.

Chapter 3

Convergence of blanket times

In this chapter, we establish asymptotic bounds on the distribution of the ε-blanket times of random walks in sequences of finite connected graphs. The precise nature of these bounds ensures convergence of the ε-blanket times of the random walks if the ε-blanket time of the limiting diffusion is continuous with probability one at ε. In Section 3.1, we introduce Assumption 1, which encodes the information that, properly rescaled, the sequences of the discrete state spaces, invariant measures, random walks, and local times, converge to (K, d_K), π, X and $(L_t(x))_{x\in K, t\in[0,T]}$ respectively, for some $T > 0$. This formulation will be described in terms of the extended Gromov-Hausdorff topology. In Section 3.2, we present Assumption 2, a weaker sufficient assumption when the sequence of spaces is equipped with resistance metrics. In Section 3.3, we prove Theorem 3.1.2 and Corollary 3.1.2.1 under Assumption 1.

3.1 Finite graphs and their associated random walks

We continue by introducing the graph theoretic framework in which we work. Firstly, let $G = (V(G), E(G))$ be a finite connected graph with at least two vertices, where $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. We endow the edge set with a symmetric weight function $\mu^G : V(G)^2 \to \mathbb{R}_+$ that satisfies $\mu^G_{xy} > 0$ if and only if $\{x, y\} \in E(G)$. The weighted random walk associated with (G, μ^G) is the Markov chain $((X^G_t)_{t\geq 0}, \mathbf{P}^G_x, x \in V(G))$ with transition probabilities $(P_G(x,y))_{x,y\in V(G)}$ given by

$$P_G(x,y) := \frac{\mu^G_{xy}}{\mu^G_x},$$

where $\mu^G_x = \sum_{y\in V(G)} \mu^G_{xy}$. One can easily check that this Markov chain is reversible and has stationary distribution given by

$$\pi^G(A) := \frac{\sum_{x\in A}\mu^G_x}{\sum_{x\in V(G)}\mu^G_x},$$

for every $A \subseteq V(G)$. The process X^G has corresponding normalized local times $(L^G_t(x))_{x\in V(G),t\geq 0}$ given by $L^G_0(x) = 0$, for every $x \in V(G)$, and, for $t \geq 1$,

$$L^G_t(x) := \frac{1}{\mu^G_x}\sum_{i=0}^{t-1} \mathbf{1}_{\{X^G_i = x\}}, \tag{3.1}$$

for every $x \in V(G)$.

The simple random walk on G is a Markov chain with transition probabilities $(P(x,y))_{x,y\in V(G)}$ given by

$$P(x,y) := 1/\deg(x),$$

where $\deg(x) = |\{y \in V(G) : y \sim x\}|$. The simple random walk is reversible and has stationary distribution given by

$$\pi(A) := \frac{\sum_{x\in A}\deg(x)}{2|E(G)|},$$

for every $A \subseteq V(G)$. It has corresponding local times as in (3.1) normalized by $\deg(x)$.

To endow G with a metric, we can choose d_G to be the shortest path distance, which collects the total weight accumulated in the shortest path between a pair of vertices in G. But this is not the most convenient choice in many cases. Another typical graph distance that arises from the view of G as an electrical network equipped with conductances $(\mu^G_{xy})_{\{x,y\}\in E(G)}$ is the so-called resistance metric.

For $f, g : V(G) \to \mathbb{R}$, let

$$\mathcal{E}_G(f, g) := \frac{1}{2} \sum_{\substack{x,y \in V(G): \\ \{x,y\} \in E(G)}} (f(x) - f(y))(g(x) - g(y))\mu^G_{xy} \tag{3.2}$$

denote the Dirichlet form associated with the process X^G. Note that the sum in the expression above counts each edge twice. One can give the following interpretation of $\mathcal{E}_G(f, f)$ in terms of electrical networks. Given a voltage f on the network, the current flow I associated with f is defined as $I_{xy} := \mu^G_{xy}(f(x) - f(y))$, for every $\{x, y\} \in E(G)$. Then, the energy dissipation of a wire connecting x and y is $\mu^G_{xy}(f(x) - f(y))^2$. So, $\mathcal{E}_G(f, f)$ is the total energy dissipation of G. We define the resistance operator on disjoint sets $A, B \in V(G)$ through the formula

$$R_G(A, B)^{-1} := \inf\{\mathcal{E}_G(f, f) : f : V(G) \to \mathbb{R}, f|_A = 0, f|_B = 1\}. \tag{3.3}$$

Now, the distance on the vertices of G defined by $R_G(x, y) := R_G(\{x\}, \{y\})$, for $x \neq y$, and $R_G(x, x) := 0$ is indeed a metric on the vertices of G. For a proof and a treatise on random walks on electrical networks see [81, Chapter 9]. We also refer the reader to [14, Section 4].

Writing τ^G_{cov} for the first time at which every vertex of G has been visited, $\mathbf{E}_x \tau^G_{\text{cov}}$ denotes the mean of this quantity when the random walk starts at $x \in V(G)$. Define the cover time by

$$t^G_{\text{cov}} := \max_{x \in V(G)} \mathbf{E}_x \tau^G_{\text{cov}}.$$

For some $\varepsilon \in (0, 1)$, define the ε-blanket time variable by

$$\tau^G_{\text{bl}}(\varepsilon) := \inf\{t \geq 0 : m(G) L^G_t(x) \geq \varepsilon t, \ \forall x \in V(G)\}, \tag{3.4}$$

where $m(G)$ is the total mass of the graph with respect to the measure μ^G, i.e. $m(G) := \sum_{x \in V(G)} \mu^G_x$. Taking the mean over the random walk started from the worst possible vertex defines the ε-blanket time, i.e.

$$t^G_{\text{bl}}(\varepsilon) := \max_{x \in V(G)} \mathbf{E}_x \tau^G_{\text{bl}}(\varepsilon).$$

Theorem 3.1.1 (Ding, Lee, Peres [45]). *For any finite connected graph $G =$*

$(V(G), E(G))$ with at least two vertices and any $\varepsilon \in (0,1)$, using the notation $\asymp$ to denote equivalence up to universal constant factors and $\asymp_\varepsilon$ to denote equivalence up to universal constant factors that depend on ε,

$$t_{\mathrm{cov}}^G \asymp |E(G)| \left(\mathbf{E} \max_{x \in V(G)} \eta_x \right)^2 \asymp_\varepsilon \tau_{\mathrm{bl}}^G(\varepsilon), \tag{3.5}$$

where $(\eta_x)_{x \in V(G)}$ is a centered Gaussian process with $\eta_{x_0} = 0$, for some $x_0 \in V(G)$, and

$$\left(\mathbf{E}(\eta_x - \eta_y)^2\right)_{x,y \in V(G)} = (R_G(x,y))_{x,y \in V(G)}.$$

Secondly, let (K, d_K) be a compact metric space and let π be a Borel probability measure of full support on (K, d_K). Take $((X_t)_{t \geq 0}, \mathbf{P}_x, x \in K)$ to be a π-symmetric Hunt process that admits local times $(L_t(x))_{x \in K, t \geq 0}$ continuous at x, uniformly over compact time intervals in t, $\mathbf{P}_x$-a.s. for every $x \in K$. Recall that a Hunt process is a strong Markov process with right-continuous sample paths that possess left limits (for definitions and other properties, see [53, Appendix A.2]). Analogously to (3.4), it is possible to define the ε-blanket time variable of K as

$$\tau_{\mathrm{bl}}(\varepsilon) := \inf\{t \geq 0 : L_t(x) \geq \varepsilon t, \ \forall x \in K\}, \tag{3.6}$$

and check that is a non-trivial quantity, see Proposition 3.3.1 below.

The following assumption encodes the information that, properly rescaled, the discrete state spaces, invariant measures, random walks, and local times, converge to (K, d_k), π, X, and $(L_t(x))_{x \in K, t \in [0,T]}$ respectively, for some fixed $T > 0$. This formulation will be described in terms of the extended Gromov-Hausdorff topology constructed in Section 2.3.

Assumption 1. *Fix $T > 0$. Let $(G^n)_{n \geq 1}$ be a sequence of finite connected graphs that have at least two vertices, for which there exist sequences of real numbers $(\alpha(n))_{n \geq 1}$ and $(\beta(n))_{n \geq 1}$, such that*

$$\left(a(n)G^n, \pi^n, (X^n_{\beta(n)t})_{t \in [0,T]}, (L^n_{\beta(n)t}(x))_{x \in V(G^n), t \in [0,T]}\right) \to \left(K, \pi, X, (L_t(x))_{x \in K, t \in [0,T]}\right)$$

in the sense of the extended pointed Gromov-Hausdorff topology, where

$$(a(n)G^n)_{n \geq 1} = (V(G^n), \alpha(n) d_{G^n}, \varrho^n)_{n \geq 1} \text{ and } K = (K, d_K, \varrho)$$

for distinguished points ϱ^n and ϱ in $V(G^n)$ and K respectively. In the above expression the definition of the discrete local times is extended to all positive times by linear interpolation.

In the examples that will be discussed in Chapter 4, we will consider random graphs. In this context, we want to verify that the previous convergence holds in distribution. Our first conclusion is the following. Its proof will be given later in Section 3.3.

Theorem 3.1.2. *Suppose that Assumption 1 holds in such a way that the time and space scaling factors satisfy $\alpha(n)\beta(n) = m(G^n)$, for every $n \geq 1$. Then, for every $\varepsilon \in (0,1)$, $\delta \in (0,1)$ and $t \in [0,T]$,*

$$\limsup_{n\to\infty} \mathbf{P}^n_{\varrho^n} \left(\beta(n)^{-1}\tau^n_{\mathrm{bl}}(\varepsilon) \leq t\right) \leq \mathbf{P}_\varrho \left(\tau_{\mathrm{bl}}(\varepsilon(1-\delta)) \leq t\right), \tag{3.7}$$

$$\liminf_{n\to\infty} \mathbf{P}^n_{\varrho^n} \left(\beta(n)^{-1}\tau^n_{\mathrm{bl}}(\varepsilon) \leq t\right) \geq \mathbf{P}_\varrho \left(\tau_{\mathrm{bl}}(\varepsilon(1+\delta)) < t\right), \tag{3.8}$$

where $\mathbf{P}^n_{\varrho^n}$ and $\mathbf{P}_\varrho$ are the laws of X^n started at ϱ^n and X started at ϱ respectively.

The mapping $\varepsilon \mapsto \tau_{\mathrm{bl}}(\varepsilon)$ is increasing in $(0,1)$, so it posseses left-hand and right-hand limits. If $\tau_{\mathrm{bl}}(\varepsilon)$ is continuous with probability 1 at ε, then letting $\delta \to 0$ on both (3.7) and (3.8) demonstrates the corollary below.

Corollary 3.1.2.1. *Suppose that Assumption 1 holds in such a way that the time and space scaling factors satisfy $\alpha(n)\beta(n) = m(G^n)$, for every $n \geq 1$. Then, for every $\varepsilon \in (0,1)$,*

$$\beta(n)^{-1}\tau^n_{\mathrm{bl}}(\varepsilon) \to \tau_{\mathrm{bl}}(\varepsilon)$$

in distribution, if $\tau_{\mathrm{bl}}(\varepsilon)$ is continuous with probability 1 at ε on $(0,1)$.

3.2 Local time convergence

To check that Assumption 1 holds we need to verify that the convergence of the local times in (2.27), as suggested by Lemma 2.3.2. Due to work done in a more general framework in [38], we can weaken the local convergence statement of (2.27) and replace it by the equicontinuity condition of (2.31). In (3.3), we defined a resistance metric on a graph viewed as an electrical network. Next, we give the definition of a regular resistance form and its associated resistance metric for arbitrary non-empty sets, which is a combination of [38, Definition 2.1] and [38, Definition 2.2].

Definition 3.2.1 (regular resistance form)**.** *Let K be a non-empty set. A pair $(\mathcal{E}, \mathcal{K})$ is called a regular resistance form on K if the following six conditions are satisfied.*

i) $\mathcal{K}$ is a linear subspace of the collection of functions $\{f : K \to \mathbb{R}\}$ containing constants, and $\mathcal{E}$ is a non-negative symmetric quadratic form on $\mathcal{K}$ such that $\mathcal{E}(f, f) = 0$ if and only if f is constant on K.

ii) Let $\sim$ be an equivalence relation on $\mathcal{K}$ defined by saying $f \sim g$ if and only if the difference $f - g$ is constant on K. Then, $(\mathcal{K}/\sim, \mathcal{E})$ is a Hilbert space.

iii) If $x \neq y$, there exists $f \in \mathcal{K}$ such that $f(x) \neq f(y)$.

iv) For any $x, y \in K$,

$$R(x, y) := \sup \left\{ \frac{|f(x) - f(y)|^2}{\mathcal{E}(f, f)} : f \in \mathcal{K},\ \mathcal{E}(f, f) > 0 \right\} < \infty. \tag{3.9}$$

v) If $\bar{f} := (f \wedge 1) \vee 0$, then $f \in \mathcal{K}$ and $\mathcal{E}(\bar{f}, \bar{f}) \leq \mathcal{E}(f, f)$, for any $f \in \mathcal{K}$.

vi) The $\mathcal{K} \cap C_0(K)$ is dense in $C_0(K)$ with respect to the supremum norm on K, where $C_0(K)$ denotes the space of compactly supported, continuous (with respect to R) functions on K.

It is the first five conditions that have to be satisfied in order for the pair $(\mathcal{E}, \mathcal{K})$ to define a resistance form. If in addition the sixth condition is satisfied then $(\mathcal{E}, \mathcal{K})$ defines a regular resistance form. Note that the fourth condition can be rewritten as $R(x, y)^{-1} = \inf\{\mathcal{E}(f, f) : f : K \to \mathbb{R}, f(x) = 0, f(y) = 1\}$, and it can be proven that it is actually a metric on K, see [74, Proposition 3.3]. It also clearly resembles the effective resistance on $V(G)$ as defined in (3.3). More specifically, taking $\mathcal{K} := \{f : V(G) \to \mathbb{R}\}$ and $\mathcal{E}_G$ as defined in (3.2) one can prove that the pair $(\mathcal{E}_G, \mathcal{K})$ satisfies the six conditions of Definition 3.2.1, and therefore is a regular resistance form on $V(G)$ with associated resistance metric given by (3.3). For a detailed proof of this fact, see [53, Example 1.2.5]. Finally, in this setting given a regular Dirichlet form, standard theory gives us the existence of an associated Hunt process $X = ((X_t)_{t\geq 0}, \mathbf{P}_x, x \in K)$ that is defined uniquely everywhere, see [53, Theorem 7.2.1] and [74, Theorem 9.9].

Suppose that the discrete state spaces $(V(G^n))_{n\geq 1}$ are equipped with resistances $(R_{G^n})_{n\geq 1}$ as defined in (3.3) and that the limiting non-empty metric space K, that appears in Assumption 1, is equipped with a resistance metric R as in Definition 3.2.1, such that

- (K, R) is compact,
- π is a Borel probability measure of full support on (K, R),
- $X = ((X_t)_{t\geq 0}, \mathbf{P}_x, x \in K)$ admits local times $L = (L_t(x))_{x\in K, t\geq 0}$ continuous at x, uniformly over compact intervals in t, $\mathbf{P}_x$ -a.s. for every $x \in K$.

In the following extra assumption we input the information encoded in the first three conclusions of Lemma 2.3.2, given that we work in a probabilistic setting instead. For simplicity as before we identify the various objects with their embeddings.

Assumption 2. *Fix $T > 0$. Let $(G^n)_{n\geq 1}$ be a sequence of finite connected graphs that have at least two vertices, for which there exist sequences of real numbers $(\alpha(n))_{n\geq 1}$ and $(\beta(n))_{n\geq 1}$, such that*

$$\left(\left(V(G^n), \alpha(n)R_{G^n}, \varrho^n\right), \pi^n, \left(X^n_{\beta(n)t}\right)_{t\in[0,T]}\right) \longrightarrow \left(\left(K, R, \varrho\right), \pi, X\right)$$

in the sense of the extended pointed Gromov-Hausdorff topology, where $\varrho^n \in V(G^n)$ and $\varrho \in K$ are distinguished points. Furthermore, suppose that for every $\varepsilon > 0$ and $T > 0$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{x\in V(G^n)} \mathbf{P}^n_x\left(\sup_{\substack{y,z\in V(G^n):\\ R_{G^n}(y,z)<\delta}} \sup_{t\in[0,T]} \alpha(n)|L^n_{\beta(n)t}(y) - L^n_{\beta(n)t}(z)| \geq \varepsilon\right) = 0. \tag{3.10}$$

It is Assumption 2 we have to verify in the examples of random graphs we will consider later. As we prove below in the last lemma of this section, if Assumption 2 holds, then the finite dimensional distributions of the local times converge, see (2.27). Given that $(V(G^n), R_{G^n})_{n\geq 1}$ and (K, R) can be isometrically embedded into a common metric space (F, d_F) such that X^n under $\mathbf{P}^n_{\varrho^n}$ converges weakly to the law of X under $\mathbf{P}_\varrho$ on $D([0,T],F)$, see Lemma 2.3.2, we can couple X^n started from ϱ^n and X started from ϱ into a common probability space such that $(X^n_{\beta(n)t})_{t\in[0,T]} \to (X_t)_{t\in[0,T]}$ in $D([0,T],F)$, almost-surely. Denote by $\mathbf{P}$ the joint probability measure under which the convergence above holds. Proving the convegence of finite dimensional distributions of local times is then an application of three lemmas that appear in [38], which we summarize below.

Lemma 3.2.1 (**Croydon, Hambly, Kumagai** [38])**.** *For every $x \in F$, $\delta > 0$, introduce the function $f_{\delta,x}(y) := \max\{0, \delta - d_F(x,y)\}$. Then, under Assumption 2,*

i) $\mathbf{P}$*-a.s., for each $x \in K$ and $T > 0$, as $\delta \to 0$,*

$$\sup_{t\in[0,T]} \left| \frac{\int_0^t f_{\delta,x}(X_s)ds}{\int_K f_{\delta,x}(y)\pi(dy)} - L_t(x) \right| \to 0.$$

ii) $\mathbf{P}$*-a.s., for each $x \in K$, $T > 0$ and $\delta > 0$, as $n \to \infty$,*

$$\sup_{t\in[0,T]} \left| \frac{\int_0^t f_{\delta,x}(X_s)ds}{\int_K f_{\delta,x}(y)\pi(dy)} - \frac{\int_0^t f_{\delta,x}(X^n_{\beta(n)s})ds}{\int_{V(G^n)} f_{\delta,x}(y)\pi^n(dy)} \right| \to 0.$$

iii) For each $x \in K$ and $T > 0$, if $x^n \in V(G^n)$ is such that $d_F(x^n, x) \to 0$, as $n \to \infty$, then

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbf{P}\left(\sup_{t\in[0,T]} \left| \frac{\int_0^t f_{\delta,x}(X^n_{\beta(n)s})ds}{\int_{V(G^n)} f_{\delta,x}(y)\pi^n(dy)} - \alpha(n)L^n_{\beta(n)t}(x^n) \right| > \varepsilon \right) = 0.$$

By applying the conclusions of Lemma 3.2.1, one deduces that for any $x \in K$ and $T > 0$, if $x^n \in V(G^n)$ such that $d_F(x^n, x) \to 0$, as $n \to \infty$, then $(\alpha(n)L^n_{\beta(n)t}(x^n))_{t\in[0,T]} \to (L_t(x))_{t\in[0,T]}$ in $\mathbf{P}$-probability in $C([0,T],\mathbb{R})$. This result extends to finite collections of points, and this is enough to establish the convergence of the finite dimensional distributions of the local times.

Lemma 3.2.2. *Suppose that Assumption 2 holds. Then, if the finite collections $(x_i^n)_{i=1}^k$ in $V(G^n)$, for $n \geq 1$, are such that $d_F(x_i^n, x_i) \to 0$, as $n \to \infty$, for some*

$(x_i)_{i=1}^k$ *in* K*, then it holds that*

$$(\alpha(n)L^n_{\beta(n)t}(x^n_i))_{i=1,\dots,k,t\in[0,T]} \to (L_t(x_i))_{i=1,\dots,k,t\in[0,T]}, \tag{3.11}$$

in distribution in $C([0,T],\mathbb{R}^k)$*.*

3.3 Blanket time-scaling and distributional bounds

In this section, under Assumption 1, and as a consequence of the local time convergence in (3.11), we are able to establish asymptotic bounds on the distribution of the blanket times of the graphs in the sequence. The same argument for the cover time-scaling was provided first in [35, Corollary 7.3] by restricting to the unweighted Sierpiński gasket graphs. The argument is applicable to any other model as long as the relevant assumptions are satisfied. First, let us check that the ε-blanket time variable of K as written in (3.6) is well-defined.

Proposition 3.3.1. *Fix* $\varepsilon \in (0,1)$*. For every* $x \in K$*,* $\mathbf{P}_x$*-a.s. we have that* $\tau_{\mathrm{bl}}(\varepsilon) \in (0,\infty)$*.*

Proof. Fix $x,y \in K$. There is a strictly positive $\mathbf{P}_x$-probability that $L_t(x) > 0$ for t large enough, which is a consequence of [85, Lemma 3.6]. From the joint continuity of local times, there exist $r \equiv r(x) > 0$, $\delta \equiv \delta(x) > 0$ and $t_* \equiv t_*(x) < \infty$, such that

$$\mathbf{P}_x\left(\inf_{z\in B_K(x,r)} L_{t_*}(z) > \delta\right) > 0. \tag{3.12}$$

Now, set $\tau_{x,y}(t_*) := \inf\{t > t_* + \tau_x : X_t = y\}$, where $\tau_x := \inf\{t > 0 : X_t = x\}$ is the hitting time of $x \in K$. In other words, $\tau_{x,y}(t_*)$ is the first hitting of $y \in K$ after $t_* + \tau_x$. Note that, the commute time identity for a resistance derived in the proof of [38, Lemma 2.9], see also Appendix B, lets it be deduced that

$$\mathbf{E}_x\tau_y \le \mathbf{E}_x\tau_y + \mathbf{E}_y\tau_x = R(x,y)\pi(K), \tag{3.13}$$

which in turn implies that $\mathbf{E}_x\tau_y < \infty$, for every $x,y \in K$. Applying this observation about the finite first moments of hitting times, it is easy to check that $\tau_{x,y}(t_*)$

is finite, $\mathbf{P}_y$-a.s., and also that

$$\mathbf{P}_y\left(\inf_{z\in B_K(x,r)} L_{\tau_{x,y}(t_*)}(z)>\delta\right)>0. \tag{3.14}$$

This simply follows from an application of (3.12) and the Strong Markov property. The additivity of local times and the Strong Markov property implies that

$$\liminf_{t\to\infty}\inf_{z\in B_K(x,r)}\frac{L_t(z)}{t}\geq\left(\sum_{i=1}^{\infty}\xi_i^1\right)\left(\sum_{i=1}^{\infty}\xi_i^2\right)^{-1}, \tag{3.15}$$

where $(\xi_i^1)_{i\geq 1}$ are independent random variables distributed as $\inf_{z\in B_K(x,r)} L_{\tau_{x,y}(t_*)}(z)$ and $(\xi_i^2)_{i\geq 1}$ are independent copies of $\tau_{x,y}(t_*)$. The strong law of large numbers along with (3.14) yields that the right-hand side of the inequality above converges to

$$\mathbf{E}_y\left[\inf_{z\in B_K(x,r)} L_{\tau_{x,y}(t_*)}(z)\right]\left(\mathbf{E}_y\tau_{x,y}(t_*)\right)^{-1},$$

$\mathbf{P}_y$-a.s. Using basic properties of the resolvents of killed processes of resistance forms (e.g. [36, (6)-(8), p. 1945]), and the commute time identity in (3.13),

$$\mathbf{E}_yL_{\tau_{x,y}(t_*)}(x)(\mathbf{E}_y\tau_{x,y}(t_*))^{-1}=\mathbf{E}_xL_{\tau_y}(x)(\mathbf{E}_y\tau_{x,y}(t_*))^{-1}=1,$$

and therefore, the joint continuity of local times lets it be deduced that the right-hand side of (3.15) satisfies

$$\liminf_{t\to\infty}\inf_{z\in B_K(x,r)}\frac{L_t(z)}{t}\geq\varepsilon_*, \tag{3.16}$$

$\mathbf{P}_y$-a.s., for some $\varepsilon_*\in(\varepsilon,1)$.

To extend this statement that holds uniformly over $B_K(x,r)$, we use the compactness of K. Consider the open cover $(B_K(x,r))_{x_i\in K}$ for K, which admits a finite subcover $(B_K(x_i,r_i))_{i=1}^N$. Since the left-hand side of (3.16) is greater than ε_*, the result clearly follows as

$$\lim_{t\to\infty}\frac{L_t(x)}{t}=\min_{1\leq i\leq N}\lim_{t\to\infty}\inf_{x\in B(x_i,r_i)}\frac{L_t(x)}{t},$$

which implies that there exists $t_0\equiv t_0(x)<\infty$, such that $L_{t_0}(x)\geq\varepsilon t_0$, for every $x\in K$, and recalling the definition of the ε-blanket time variable of K in (3.6), we deduce that $\tau_{\mathrm{bl}}(\varepsilon) \leq t_0 < \infty$.

$\square$

We are now ready to prove one of our main results.

Proof of Theorem 3.1.2. Let $\varepsilon \in (0,1)$, $\delta \in (0,1)$ and $t \in [0,T]$. Suppose that $t < \tau_{\mathrm{bl}}(\varepsilon(1-\delta))$. Then, there exists a $y \in K$ for which $L_t(y) < \varepsilon(1-\delta)t$. Using the Skorohod representation theorem, we can assume that the conclusions of Lemma 2.3.2 hold in an almost-sure sense. From (2.24), there exists $y^n \in V(G^n)$ such that, for n large enough, $d_F(y^n, y) < 2\varepsilon$. Then, the local convergence at (2.27) implies that, for n large enough,

$$\alpha(n) L^n_{\beta(n)t}(y^n) \leq L_t(y) + \varepsilon\delta t.$$

Thus, for n large enough, it follows that $\alpha(n) L^n_{\beta(n)t}(y^n) \leq L_t(y) + \varepsilon\delta t < \varepsilon t$. Using the time and space scaling identity, we deduce $m(G^n) L^n_{\beta(n)t}(y^n) < \varepsilon\beta(n)t$, for n large enough, which in turn implies that $\beta(n)t \leq \tau^n_{\mathrm{bl}}(\varepsilon)$, for n large enough. As a consequence, we get that $\tau_{\mathrm{bl}}(\varepsilon(1-\delta)) \leq \liminf_{n\to\infty} \beta(n)^{-1}\tau^n_{\mathrm{bl}}(\varepsilon)$, which proves (3.7).

Assume now that $\tau_{\mathrm{bl}}(\varepsilon(1+\delta)) < t$. Then, for some $\tau_{\mathrm{bl}}(\varepsilon(1+\delta)) \leq t_0 < t$, it is the case that $L_{t_0}(y) \geq \varepsilon(1+\delta)t_0$, for every $y \in K$. As in the previous paragraph, using the Skorohod represantation theorem, we suppose that the conclusions of Lemma 2.3.2 hold almost-surely. From (2.24), for every $y^n \in V(G^n)$, there exists a $y \in K$ such that, for n large enough, $d_F(y^n, y) < 2\varepsilon$. From the local convergence at (2.27) we have that, for n large enough,

$$\alpha(n) L^n_{\beta(n)t_0}(y^n) \geq L_{t_0}(y) - \varepsilon\delta t_0.$$

Therefore, for n large enough, it follows that $\alpha(n) L^n_{\beta(n)t_0}(y^n) \geq L_{t_0}(y) - \varepsilon\delta t_0 \geq \varepsilon t_0$, for every $y \in K$. As before, using the time and space scaling identity yields $m(G^n) L^n_{\beta(n)t_0}(y^n) \geq \varepsilon\beta(n)t_0$, for every $y^n \in V(G^n)$ and large enough n, which in turn implies that $\beta(n)t_0 \geq \tau^n_{\mathrm{bl}}(\varepsilon)$, for n large enough. As a consequence, we get that $\limsup_{n\to\infty} \beta(n)^{-1}\tau^n_{\mathrm{bl}}(\varepsilon) \leq \tau_{\mathrm{bl}}(\varepsilon(1+\delta))$, from which (3.8) follows.

$\square$

Chapter 4

Examples

In this chapter, we demonstrate that it is possible to apply our main result to a number of examples where the graphs and the limiting spaces are random. These examples include critical Galton-Watson trees, the critical Erdős-Rényi random graph and the critical regime of the configuration model. The aforementioned models of sequences of random graphs exhibit a mean-field behavior at criticality in the sense that the scaling exponents for the walks, and consequently for the local times, are a multiple of the volume and the diameter of the graphs. In the first few pages of each section we quickly survey some of the key features of each example that will be helpful when verifying Assumption 2. Our method used in proving continuity of the blanket time of the limiting diffusion is generic in the sense that it applies on each random metric measure space and a corresponding σ-finite measure that generates realizations of the random metric measure space in such a way that rescaling the σ-finite measure by a constant factor results in generating the same space with its metric and measure perturbed by a multiple of this factor. For that reason we believe our results to easily transfer when considering Galton-Watson trees with critical offspring distribution in the domain of attraction of a stable law with index $\alpha \in (1, 2)$, see [78, Theorem 4.3] and random stable looptrees, see [40, Theorem 4.1]. Also, we hope our work to be seen as a stepping stone to deal with the more delicate problem of establishing convergence in distribution of the rescaled cover times of the discrete-time walks in each application of our main result. See [35, Remark 7.4] for a thorough discussion on the demanding nature of this project.

To demonstrate our main results consider first T, a critical Galton-Watson tree (with finite variance σ^2). The following result on the cover time of the simple random walk was obtained by Aldous, see [5, Proposition 15], which we apply to the blanket time in place of the cover time. The two parameters are equivalent up to universal constants, see (3.5).

Theorem 4.0.1 (**Aldous** [5]). *Let T be a critical Galton-Watson tree (with finite variance σ^2). For any $\delta > 0$ there exists $A = A(\delta, \varepsilon, \sigma^2) > 0$, such that*

$$P(A^{-1}k^{3/2} \leq t_{\mathrm{bl}}^{T}(\varepsilon) \leq Ak^{3/2} | |T| \in [k, 2k]) \geq 1 - \delta,$$

for every $\varepsilon \in (0, 1)$.

Now, let $G(n, p)$ be the resulting subgraph of the complete graph on n vertices obtained by p-bond percolation. If $p = n^{-1} + \lambda n^{-4/3}$, for some $\lambda \in \mathbb{R}$, that is when we are in the so-called critical window, the largest connected component $\mathcal{C}_1^n$, as a graph, converges to a random compact metric space $\mathcal{M}$ that can be constructed directly from the Brownian CRT $\mathcal{T}_e$, see the work of [3]. The following result on the blanket time of the simple random walk on $\mathcal{C}_1^n$ is due to Barlow, Ding, Nachmias and Peres [16].

Theorem 4.0.2 (**Barlow, Ding, Nachmias, Peres** [16]). *Let $\mathcal{C}_1^n$ be the largest connected component of $G(n, p)$, $p = n^{-1} + \lambda n^{-4/3}$, $\lambda \in \mathbb{R}$ fixed. For any $\delta > 0$ there exists $B = B(\delta, \varepsilon) > 0$, such that*

$$P(B^{-1}n \leq t_{\mathrm{bl}}^{\mathcal{C}_1^n}(\varepsilon) \leq Bn) \geq 1 - \delta,$$

for every $\varepsilon \in (0, 1)$.

Our contribution refines the previous existing tightness results on the order of the blanket time. In what follows $\mathbb{P}_{\varrho^n}$, $n \geq 1$ as well as $\mathbb{P}_{\varrho}$ are the annealed measures, that is the probability measures obtained by integrating out the randomness of the state spaces involved.

Theorem 4.0.3. *Let $\mathcal{T}_n$ be a critical Galton-Watson tree (with finite variance) conditioned to have total progeny $n + 1$. Fix $\varepsilon \in (0, 1)$. If $\tau_{\mathrm{bl}}^n(\varepsilon)$ is the ε-blanket time variable of the simple random walk on $\mathcal{T}_n$, started from its root ϱ^n, then*

$$\mathbb{P}_{\varrho^n}\left(n^{-3/2}\tau_{\mathrm{bl}}^n(\varepsilon) \leq t\right) \to \mathbb{P}_{\varrho}\left(\tau_{\mathrm{bl}}^e(\varepsilon) \leq t\right),$$

for every $t \geq 0$, where $\tau_{\mathrm{bl}}^{e}(\varepsilon) \in (0,\infty)$ is the ε-blanket time variable of the Brownian motion on $\mathcal{T}_e$, started from a distinguished point $\varrho \in \mathcal{T}_e$. Equivalently, for every $\varepsilon \in (0,1)$, $n^{-3/2}\tau_{\mathrm{bl}}^{n}(\varepsilon)$ under $\mathbb{P}_{\varrho^n}$ converges weakly to $\tau_{\mathrm{bl}}^{e}(\varepsilon)$ under $\mathbb{P}_{\varrho}$.

Theorem 4.0.4. *Fix $\varepsilon \in (0,1)$. If $\tau_{\mathrm{bl}}^{n}(\varepsilon)$ is the ε-blanket time variable of the simple random walk on $\mathcal{C}_1^n$, started from its root ϱ^n, then*

$$\mathbb{P}_{\varrho^n}\left(n^{-1}\tau_{\mathrm{bl}}^{n}(\varepsilon) \leq t\right) \to \mathbb{P}_{\varrho}\left(\tau_{\mathrm{bl}}^{\mathcal{M}}(\varepsilon) \leq t\right),$$

for every $t \geq 0$, where $\tau_{\mathrm{bl}}^{\mathcal{M}}(\varepsilon) \in (0,\infty)$ is the ε-blanket time variable of the Brownian motion on $\mathcal{M}$, started from ϱ.

To present our last result we consider the configuration model. Let $\mathrm{M}^n(d)$ be the random multigraph labeled by $[n] := \{1,2,...,n\}$, such that the i-th vertex has degree d_i, $i \geq 1$, for every $1 \leq i \leq n$, which is constructed as follows. Assign d_i half-edges to each vertex i, labelling them in an arbitrary way. Then, the configuration model is produced by a uniform pairing of the half-edges to create full edges. If the degree sequence satisfies certain conditions that would be made precise later in Assumption 3, it was shown in the work of [42] that the largest connected component $\mathrm{M}_1^n(d)$, is of order $n^{2/3}$. Recently in [23] its scaling limit, $\mathcal{M}_D$, was proven to exist and to belong to the Erdős-Rényi universality class.

Theorem 4.0.5. *Fix $\varepsilon \in (0,1)$. If $\tau_{\mathrm{bl}}^{n}(\varepsilon)$ is the ε-blanket time variable of the simple random walk on $\mathrm{M}_1^n(d)$, started from its root ϱ^n, then*

$$\mathbb{P}_{\varrho^n}\left(n^{-1}\tau_{\mathrm{bl}}^{n}(\varepsilon) \leq t\right) \to \mathbb{P}_{\varrho}\left(\tau_{\mathrm{bl}}^{\mathcal{M}_D}(\varepsilon) \leq t\right),$$

for every $t \geq 0$, where $\tau_{\mathrm{bl}}^{\mathcal{M}_D}(\varepsilon) \in (0,\infty)$ is the ε-blanket time variable of the Brownian motion on $\mathcal{M}_D$, started from ϱ.

In each section that comprises Chapter 4, we verify the assumptions of Corollary 3.1.2.1, and therefore prove convergence of blanket times for the series of critical random graphs mentioned above, thus effectively proving Theorem 4.0.3-4.0.5.

4.1 Critical Galton-Watson trees

We start by briefly describing the connection between critical Galton-Watson trees and the Brownian CRT. Let ξ be a mean 1 random variable with variance $0 < \sigma_\xi^2 < +\infty$, whose distribution is aperiodic (its support generates the lattice $\mathbb{Z}$, not just a strict subgroup of $\mathbb{Z}$). Let $\mathcal{T}_n$ be a Galton-Watson tree with offspring distribution ξ conditioned to have total number of vertices $n+1$, which is well-defined for every n large enough from the aperiodicity of the distribution of ξ. Then, it is the case that

$$\left(V(\mathcal{T}_n), n^{-1/2} d_{\mathcal{T}_n}\right) \to (\mathcal{T}_e, d_e)\,, \tag{4.1}$$

in distribution with respect to the Gromov-Hausdorff distance between compact metric spaces, where $d_{\mathcal{T}_n}$ is the shortest path distance on the graph with vertex set $V(\mathcal{T}_n)$, see [6] and [77].

To describe the limiting object in (4.1), let $e := (e(t))_{0\leq t\leq 1}$ denote the normalized Brownian excursion, which in the narrow sense is a linear Brownian motion, started from zero, conditioned to remain positive in $(0,1)$ and come back to zero at time 1. The process corresponding to this intuitive description was characterized explicitly in several ways in Section 2.2. We extend the definition of e by setting $e(t) = 0$, if $t > 1$. Recalling the notion of compact real trees coded by functions from Section 2.1,

$$(\mathcal{T}_e, d_e) = ([0,1]/\sim, d_e)$$

is the Brownian CRT, cf. (2.2) and (2.3). The natural Borel probability measure $\mu_{\mathcal{T}_e}$ upon $\mathcal{T}_e$ is the image measure on $\mathcal{T}_e$ of the Lebesgue measure ℓ on $[0,1]$ by the canonical projection p_e of $[0,1]$ onto $\mathcal{T}_e$, cf. (2.4).

Upon almost-every realization of the metric measure space $((\mathcal{T}_e, d_e), \mu_{\mathcal{T}_e})$, it is possible to define a corresponding Brownian motion X^e. The way this can be done is described in [31, Section 2.2]. If we denote by $\mathbf{P}_{\varrho^n}^{\mathcal{T}_n}$ the law of the simple random walk in $\mathcal{T}_n$, started from a distinguished point ϱ^n, and by π^n the stationary probability measure, then as it was shown in [33], the scaling limit in (4.1) can be extended to the distributional convergence of

$$\left(\left(V(\mathcal{T}_n), n^{-1/2} d_{\mathcal{T}_n}, \varrho^n\right), \pi^n(n^{1/2}\cdot), \mathbf{P}_{\varrho^n}^{\mathcal{T}_n}\left((n^{-1/2} X^n_{\lfloor n^{3/2} t\rfloor})_{t\in[0,1]} \in \cdot\right)\right)$$

to $\big((\mathcal{T}_e, d_e, \varrho), \mu_{\mathcal{T}_e}, \mathbf{P}^e_\varrho\big)$, where $\mathbf{P}^e_\varrho$ is the law of X^e, started from a distinguished point ϱ. This convergence described in [33] holds after embedding all the relevant objects nicely into a Banach space. We can reformulate this result in terms of the extended Gromov-Hausdorff topology that incorporates distinguished points. Namely,

$$\big((V(\mathcal{T}_n), n^{-1/2} d_{\mathcal{T}_n}, \varrho^n), \pi^n, (n^{-1/2} X^n_{\lfloor n^{3/2} t \rfloor})_{t\in[0,1]}\big) \to \big((\mathcal{T}_e, d_e, \varrho), \mu_{\mathcal{T}_e}, (X^e_{(\sigma_\xi/2)t})_{t\in[0,1]}\big), \tag{4.2}$$

in distribution in an extended pointed Gromov-Hausdorff sense.

Next, we introduce the contour function of $\mathcal{T}_n$. Informally, it encodes the trace of the motion of a particle that starts from the root at time $t = 0$ and then explores the tree from left to right, moving continuously at unit speed along its edges. Formally, we define a function first for integer arguments as follows:

$$f(0) = \varrho^n.$$

Given $f(i) = v$, we define $f(i+1)$ to be, if possible, the leftmost child that has not been visited yet, let's say w. If no child is left unvisited, we let $f(i+1)$ be the parent of v. Then, the contour function of $\mathcal{T}_n$, is defined as the distance between $f(i)$ and the root ϱ^n, i.e.

$$C_n(i) := d_{\mathcal{T}_n}(\varrho^n, f(i)), \qquad 0 \le i \le 2n.$$

The function C_n is only defined for integer arguments. To map intermediate values of f into $V(\mathcal{T}_n)$ extend f to $[0, 2n]$ by taking $f(t)$ to be $f(\lfloor t \rfloor)$ or $f(\lceil t \rceil)$, whichever is further away from the root. This convention will be used later in a calculation involved in the proof of Theorem 5.4.1. The following theorem is due to Aldous.

Theorem 4.1.1 (**Aldous** [6]). *Let $C_{(n)}$ denote the normalized contour function of $\mathcal{T}_n$, defined by*

$$C_{(n)}(s) := \frac{C_n(2ns)}{\sqrt{n}}, \qquad 0 \le s \le 1.$$

Then, the following convergence holds in distribution in $C([0,1])$: $C_{(n)} \xrightarrow{(d)} v := (2/\sigma_\xi) e$, where e is a normalized Brownian excursion.

An essential tool in what follows will be a universal concentration estimate of the fluctuations of local times that holds uniformly over compact time intervals.

For the statement of this result let

$$r(\mathcal{T}_n) := \sup_{x,y \in V(\mathcal{T}_n)} d_{\mathcal{T}_n}(x,y)$$

denote the diameter of $\mathcal{T}_n$ and $m(\mathcal{T}_n)$ denote the total mass of $\mathcal{T}_n$. Also, we introduce the rescaled shortest path distance $\tilde{d}_{\mathcal{T}_n}(x,y) := r(\mathcal{T}_n)^{-1} d_{\mathcal{T}_n}(x,y)$.

Theorem 4.1.2 (**Croydon** [35]). *For every $T > 0$, there exist constants c_1 and c_2 not depending on $\mathcal{T}_n$, such that*

$$\sup_{y,z \in V(\mathcal{T}_n)} \mathbf{P}_{\varrho^n}^{\mathcal{T}_n} \left(r(\mathcal{T}_n)^{-1} \sup_{t \in [0,T]} \left| L^n_{r(\mathcal{T}_n)m(\mathcal{T}_n)t}(y) - L^n_{r(\mathcal{T}_n)m(\mathcal{T}_n)t}(z) \right| \geq \lambda \sqrt{\tilde{d}_{\mathcal{T}_n}(y,z)} \right) \\ \leq c_1 e^{-c_2 \lambda}, \tag{4.3}$$

for every $\lambda \geq 0$. Moreover, the constants can be chosen in such a way that only c_1 depends on T.

We remark here that the product $m(\mathcal{T}_n) r(\mathcal{T}_n)$, that is the product of the volume and the diameter of the graph, which is also the maximal commute time of the random walk, gives the natural time-scaling for the various models of sequences of critical random graphs we are going to consider. The concentration estimate of Theorem 4.1.2 is a version of [26, (V.3.28)] for graphs. The last ingredient we are going to make considerable use of is the tightness of the sequence $||C_{(n)}||_{H_\alpha}$ of Hölder norms, for some $\alpha > 0$. The proof of Theorem 4.1.3 is based on Kolmogorov's continuity criterion (and its proof to get uniformity in n). Indeed, the following result can be obtained for any $\alpha \in (0, 1/2)$.

Theorem 4.1.3 (**Janson and Marckert** [64]). *There exists $\alpha > 0$, such that for every $\varepsilon > 0$ there exists a finite real number K_ε, such that*

$$P\left(\sup_{0 \leq s \neq t \leq 1} \frac{|C_{(n)}(s) - C_{(n)}(t)|}{|t-s|^\alpha} \leq K_\varepsilon \right) \geq 1 - \varepsilon,$$

uniformly on n.

Remark. *Building upon [55], Janson and Marckert proved this precise estimate on the geometry of the trees when the offspring distribution has finite exponential moments. Relaxing this strong condition to only a finite variance assumption, the* *recent work of Marzouk and more specifically [86, Lemma 1] implies that Theorem 4.1.3 holds for the normalized height function of $\mathcal{T}_n$, which constitutes an alternative encoding of the trees. That Theorem 4.1.3 can be stated as well in terms of the normalized contour function of $\mathcal{T}_n$, with only a finite variance assumption, is briefly achieved using that the normalized contour function is arbitrarily close to a time-changed normalized height process. See the equation that appears as (15) in [77, Theorem 1.7] and refer to [77, Section 1.6] for a detailed discussion.*

Since $(\mathcal{T}_n)_{n\geq 1}$ is a collection of graph trees it follows that the shortest path distance $d_{\mathcal{T}_n}$, $n \geq 1$ is identical to the resistance metric on the vertex set $V(\mathcal{T}_n)$, $n \geq 1$, where each edge has unit conductance. In this context, we make use of the full machinery provided by the theorems above in order to prove that the local times are equicontinuous with respect to the annealed law, which is formally defined for suitable events as

$$\mathbb{P}_{\varrho^n}(\cdot) := \int \mathbf{P}_{\varrho^n}^{\mathcal{T}_n}(\cdot) P(d\mathcal{T}_n).$$

Proposition 4.1.4. *For every $\varepsilon > 0$ and $T > 0$,*

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \mathbb{P}_{\varrho^n}\left(\sup_{\substack{y,z\in V(\mathcal{T}_n):\\ n^{-1/2}d_{\mathcal{T}_n}(y,z)<\delta}} \sup_{t\in[0,T]} n^{-1/2}|L^n_{n^{3/2}t}(y) - L^n_{n^{3/2}t}(z)| \geq \varepsilon \right) = 0.$$

Proof. Let us define, similarly to d_e, the distance $d_{C_{(n)}}$ in $[0,1]$ by setting:

$$d_{C_{(n)}}(t_1,t_2) := C_{(n)}(t_1) + C_{(n)}(t_2) - 2 \min_{r\in[t_1\wedge t_2, t_1\vee t_2]} C_{(n)}(r).$$

Using the terminology introduced to describe the Brownian CRT, $\mathcal{T}_n$ equipped with $n^{1/2}d_{C_{(n)}}$, when t_1 and t_2 are equivalent if and only if

$$C_{(n)}(t_1) = C_{(n)}(t_2) = \min_{r\in[t_1\wedge t_2, t_1\vee t_2]} C_{(n)}(r),$$

coincides with the tree coded by $n^{1/2}C_{(n)}$. We denote by $p_{C_{(n)}} : [0,1] \to \mathcal{T}_n$ the canonical projection that maps every time point in $[0,1]$ to its equivalence class on $\mathcal{T}_n$.

Given $t_1, t_2 \in [0,1]$, with $2nt_1$ and $2nt_2$ integers, such that $p_{C_{(n)}}(t_1) = y$ and $p_{C_{(n)}}(t_2) = z$, let $u \in [t_1 \wedge t_2, t_1 \vee t_2]$ with $\min_{r\in[t_1\wedge t_2, t_1\vee t_2]} C_{(n)}(r) = C_{(n)}(u)$. From

Theorem 4.1.3, there exist $K > 0$ and $\alpha > 0$, such that

$$\begin{aligned} d_{C_{(n)}}(t_1, t_2) &= (C_{(n)}(t_1) - C_{(n)}(u)) + (C_{(n)}(t_2) - C_{(n)}(u)) \leq K(|t_1 - u|^\alpha + |u - t_2|^\alpha) \\ &\leq 2K|t_1 - t_2|^\alpha \end{aligned} \tag{4.4}$$

with probability arbitrarily close to 1, where the last inequality follows from the concavity of t^α. We condition on $C_{(n)}$, assuming that it satisfies (4.4). The total length of the path between y and z using (4.4), is

$$C_n(2nt_1) + C_n(2nt_2) - 2 \min_{r \in [t_1 \wedge t_2, t_1 \vee t_2]} C_n(2nr) = n^{1/2} d_{C_{(n)}}(t_1, t_2) \leq 2Kn^{1/2}|t_1 - t_2|^\alpha.$$

Hence, by Theorem 4.1.2, if we denote by

$$||L^n(x)||_{\infty,[0,T]} := \sup_{t \in [0,T]} |L^n_t(x)|, \qquad x \in V(\mathcal{T}_n),$$

the supremum norm of $L^n(x) : [0, T] \to \mathbb{R}_+$, for any fixed $p \geq 2$,

$$\mathbf{E}^{\mathcal{T}_n}_{\varrho^n} \left|\left| r(\mathcal{T}_n)^{-1} \big(L^n_{r(\mathcal{T}_n)m(\mathcal{T}_n)\cdot}(y) - L^n_{r(\mathcal{T}_n)m(\mathcal{T}_n)\cdot}(z) \big) \right|\right|^p_{\infty,[0,T]}$$

$$= \int_0^\infty \mathbf{P}^{\mathcal{T}_n}_{\varrho^n} \left(\sup_{t \in [0,T]} r(\mathcal{T}_n)^{-1} \left| L^n_{r(\mathcal{T}_n)m(\mathcal{T}_n)t}(y) - L^n_{r(\mathcal{T}_n)m(\mathcal{T}_n)t}(z) \right| \geq \varepsilon^{1/p} \right) d\varepsilon$$

$$\leq c_1 \int_0^\infty e^{-c_2 \frac{\varepsilon^{1/p}}{\sqrt{r(\mathcal{T}_n)^{-1} d_{\mathcal{T}_n}(y,z)}}} d\varepsilon.$$

Changing variables, $\lambda^{1/p} = \frac{\varepsilon^{1/p}}{\sqrt{r(\mathcal{T}_n)^{-1} d_{\mathcal{T}_n}(y,z)}}$, yields

$$\begin{aligned} &\int_0^\infty e^{-c_2 \frac{\varepsilon^{1/p}}{\sqrt{r(\mathcal{T}_n)^{-1} d_{\mathcal{T}_n}(y,z)}}} d\varepsilon \\ &= (r(\mathcal{T}_n)^{-1} d_{\mathcal{T}_n}(y, z))^{p/2} \int_0^\infty e^{-c_2 \lambda^{1/p}} d\lambda \leq c_3 (r(\mathcal{T}_n)^{-1} d_{\mathcal{T}_n}(y, z))^{p/2}, \end{aligned}$$

where c_3 is a constant depending only on p. Therefore,

$$\mathbf{E}^{\mathcal{T}_n}_{\varrho^n} \left|\left| r(\mathcal{T}_n)^{-1} \big(L^n_{r(\mathcal{T}_n)m(\mathcal{T}_n)\cdot}(y) - L^n_{r(\mathcal{T}_n)m(\mathcal{T}_n)\cdot}(z) \big) \right|\right|^p_{\infty,[0,T]} \leq c_3 (r(\mathcal{T}_n)^{-1} d_{\mathcal{T}_n}(y, z))^{p/2}. \tag{4.5}$$

Conditioning on the event that $C_{(n)}$ satisfies (4.4), the total length of the path between y and z is bounded above by

$$2Kn^{1/2}|t_1 - t_2|^{\alpha},$$

and consequently the diameter of $\mathcal{T}_n$ is bounded above by a multiple of $n^{1/2}$. More specifically,

$$r(\mathcal{T}_n) \leq Kn^{1/2}2^{\alpha+1}.$$

Moreover, $m(\mathcal{T}_n) = 2E(\mathcal{T}_n) = 2(V(\mathcal{T}_n) - 1) = 2n$. Hence, by (4.5), we have shown that, conditioned on $C_{(n)}$ satisfying (4.4), for any fixed $p \geq 2$,

$$\begin{aligned}\mathbf{E}_{\varrho^n}^{\mathcal{T}_n} \left|\left|n^{-1/2}\left(L^n_{n^{3/2}\cdot}(y) - L^n_{n^{3/2}\cdot}(z)\right)\right|\right|^p_{\infty,[0,T]} &\leq c_4(n^{-1/2}d_{\mathcal{T}_n}(y,z))^{p/2}2^{-\alpha p} \\ &\leq c_5\left|t_1 - t_2\right|^{\alpha p/2}.\end{aligned}$$

Choosing p such that $\alpha p \geq 4$, this is at most, except in the trivial case $t_1 = t_2$,

$$c_5|t_1 - t_2|^2.$$

This holds for all t_1 and t_2, with $2nt_1$ and $2nt_2$ integers, such that $p_{C_{(n)}}(t_1) = y$ and $p_{C_{(n)}}(t_2) = z$. Since the discrete local time process is interpolated linearly between these time points, it also holds for every $t_1, t_2 \in [0,1]$. Using the moment condition (13.14) of [25, Theorem 13.5] yields that, on the event that $C_{(n)}$ satisfies (4.4), the sequence

$$\left|\left|n^{-1/2}L^n_{n^{3/2}\cdot}(p_n(t_1))\right|\right|_{\infty,[0,T]}$$

is tight in $C[0,1]$. If we denote by A_n^{δ} the measurable event

$$A_n^{\delta} := \left\{\sup_{\substack{y,z\in V(\mathcal{T}_n): \\ n^{-1/2}d_{\mathcal{T}_n}(y,z)<\delta}} \sup_{t\in[0,T]} n^{-1/2}|L^n_{n^{3/2}t}(y) - L^n_{n^{3/2}t}(z)| \geq \varepsilon\right\},$$

note that

$$\mathbb{P}_{\varrho^n}(A_n^\delta) = \int \mathbf{P}_{\varrho^n}^{\mathcal{T}_n}(A_n^\delta)P(d\mathcal{T}_n) \leq \int \mathbf{P}_{\varrho^n}^{\mathcal{T}_n}\left(A_n^\delta;\ \sup_{t_1,t_2\in[0,1]} \frac{d_{C_{(n)}}(t_1,t_2)}{|t_1-t_2|^\alpha} \leq 2K\right)P(d\mathcal{T}_n)$$

$$+ P\left(d_{C_{(n)}}(t_1,t_2) > 2K|t_1-t_2|^\alpha,\ \forall t_1,t_2\in[0,1]\right),$$

and therefore, as a consequence of the Reverse Fatou Lemma,

$$\limsup_{n\to\infty} \mathbb{P}_{\varrho^n}(A_n^\delta) \leq \int \limsup_{n\to\infty} \mathbf{P}_{\varrho^n}^{\mathcal{T}_n}\left(A_n^\delta;\ \sup_{t_1,t_2\in[0,1]} \frac{d_{C_{(n)}}(t_1,t_2)}{|t_1-t_2|^\alpha} \leq 2K\right)P(d\mathcal{T}_n)$$

$$+ \limsup_{n\to\infty} P\left(d_{C_{(n)}}(t_1,t_2) > 2K|t_1-t_2|^\alpha,\ \forall t_1,t_2\in[0,1]\right).$$

Letting $\delta \to 0$, the desired result follows using the tightness of the local times, conditioned on $C_{(n)}$ satisfying (4.4), which was shown before, and using the fact that the second probability on the right-hand side above, by (4.4), is arbitrarily small.

□

4.1.1 Continuity of blanket times of Brownian motion on the Brownian CRT

We are primarily interested in proving continuity of the ε-blanket time variable of the Brownian motion on the Brownian CRT. The mapping $\varepsilon \mapsto \tau_{\mathrm{bl}}^e(\varepsilon)$ is increasing in (0,1), so it posseses left-hand and right-hand limits. We let

$$\mathcal{A}_\varepsilon := \left\{(\mathcal{T}_e)_{e\in E} : \mathbf{P}_\varrho^e\left(\tau_{\mathrm{bl}}^e(\varepsilon-) = \tau_{\mathrm{bl}}^e(\varepsilon+)\right) = 1\right\} \tag{4.6}$$

denote the collection of random trees coded by positive excursions that have continuous blanket time variable at $\varepsilon \in (0,1)$ almost-surely with respect to $\mathbf{P}_\varrho^e$, the law of the corresponding Brownian motion on $\mathcal{T}_e$.

Moreover, $\varepsilon \mapsto \tau_{\mathrm{bl}}^e(\varepsilon)$ has at most a countably infinite number of discontinuities $\mathbf{P}_\varrho^e$-a.s as a real-valued monotone function defined on an interval. Recalling the definition of Itô's (unconditioned) excursion measure $\mathbb{N}$ in (2.6), by Fubini, we immediately get

$$\int_0^1 \int_E \mathbf{P}_\varrho^e \left(\tau_{\mathrm{bl}}^e(\varepsilon-) \neq \tau_{\mathrm{bl}}^e(\varepsilon+)\right) \mathbb{N}(de)d\varepsilon = \mathbf{E}_{\mathbb{P}_\varrho}\left[\int_0^1 \mathbb{1}\left\{\tau_{\mathrm{bl}}^e(\varepsilon-) \neq \tau_{\mathrm{bl}}^e(\varepsilon+)\right\} d\varepsilon\right] = 0,$$

where by $\mathbf{E}_{\mathbb{P}_\varrho}$ we denoted the expectation with respect to the annealed law, which is formally defined for suitable events as

$$\mathbb{P}_\varrho(\cdot) := \int_E \mathbf{P}_\varrho^e(\cdot)\mathbb{N}(de). \tag{4.7}$$

Therefore, denoting the Lebesgue measure on the real line by ℓ as usual, we deduce that for ℓ-a.e. $\varepsilon \in (0,1)$,

$$\int_E \left(1 - \mathbf{P}_\varrho^e \left(\tau_{\mathrm{bl}}^e(\varepsilon-) = \tau_{\mathrm{bl}}^e(\varepsilon+)\right)\right) \mathbb{N}(de) = 0.$$

The fact that $\mathbb{N}(de)$ is a sigma-finite measure on E yields that for ℓ-a.e. $\varepsilon \in (0,1)$, $\mathbb{N}$-a.e. $e \in E$,

$$\mathbf{P}_\varrho^e \left(\tau_{\mathrm{bl}}^e(\varepsilon-) = \tau_{\mathrm{bl}}^e(\varepsilon+)\right) = 1. \tag{4.8}$$

Thus, we inferred that for ℓ-a.e. $\varepsilon \in (0,1)$, $\mathbb{N}$-a.e. $e \in E$, $\mathcal{T}_e \in \mathcal{A}_\varepsilon$. To be satisfactory for our purposes, we need to improve this statement to hold globally in $(0,1)$.

For a fixed positive excursion e compactly supported on $[0,\zeta]$, consider the random real tree $((\mathcal{T}_e, d_e), \mu_{\mathcal{T}_e})$, where d_e is defined as in (2.2) and $\mu_{\mathcal{T}_e}$ as in (2.4). Recalling the mapping introduced in Section 2.2,

$$\Theta_a(e)(t) = \sqrt{a}e(t/a), \qquad t \in [0, a\zeta],$$

applied to e for some $a > 1$, results in perturbing d_e by a factor of $\sqrt{a}$ and $\mu_{\mathcal{T}_e}$ by a factor of a. To be more precise, consider the set $A \cap \mathrm{Sk}(\mathcal{T}_{\Theta_a(e)})$, see (2.1), where $A \in \mathcal{B}(\mathcal{T}_{\Theta_a(e)})$. In particular, if $\mathcal{T}' \subseteq \mathcal{T}_{\Theta_a(e)}$ is a countable dense subset, we have that

$$\left\{A \cap \mathrm{Sk}(\mathcal{T}_{\Theta_a(e)}) : A \in \mathcal{B}(\mathcal{T}_{\Theta_a(e)})\right\} = \sigma\left(\{[x,y] : x, y \in \mathcal{T}'\}\right). \tag{4.9}$$

For $s,t \in p^{-1}_{\Theta_a(e)}(\mathcal{T}') \subseteq [0, a\zeta]$, such that $p_{\Theta_a(e)}(s) = x$ and $p_{\Theta_a(e)}(t) = y$, observe that

$$d_{\Theta_a(e)}(x,y) = \sqrt{a}d_e(\tilde{x}_a, \tilde{y}_a), \tag{4.10}$$

where $\tilde{x}_a, \tilde{y}_a \in \mathcal{T}_e$ are in such a way that $p_e(s/a) = \tilde{x}_a$ and $p_e(t/a) = \tilde{y}_a$. Using the scaling property of the Lebesgue measure, implies

$$\begin{aligned}\mu_{\mathcal{T}_{\Theta_a(e)}}([x,y]) &= \ell(\{r \in [0, a\zeta] : p_{\Theta_a(e)}(r) \in [x,y]\}) \\ &= \ell(\{r \in [0, a\zeta] : p_e(r/a) \in [\tilde{x}_a, \tilde{y}_a]\}) \\ &= a\ell(\{r \in [0, \zeta] : p_e(r) \in [\tilde{x}_a, \tilde{y}_a]\}).\end{aligned}$$

Therefore,

$$\mu_{\mathcal{T}_{\Theta_a(e)}}([x,y]) = a\mu_{\mathcal{T}_e}([\tilde{x}_a, \tilde{y}_a]). \tag{4.11}$$

For simplicity, for the random real tree $\mathcal{T} := ((\mathcal{T}_e, d_e), \mu_{\mathcal{T}_e})$, we write $\Theta_a\mathcal{T}$ to denote the random real tree $((\mathcal{T}_{\Theta_a(e)}, d_{\Theta_a(e)}), \mu_{\mathcal{T}_{\Theta_a(e)}})$, where $d_{\Theta_a(e)}$ and $\mu_{\mathcal{T}_{\Theta_a(e)}}$ satisfy (4.10) and (4.11) respectively.

Next, if the Brownian motion $(X_t^e)_{t\geq 0}$ on $\mathcal{T}$ admits local times $(L_t^e(x))_{x\in\mathcal{T},t\geq 0}$ that, $\mathbf{P}_\varrho^e$-a.s., are jointly continuous in (x,t), then it is the case that the Brownian motion on $\Theta_a\mathcal{T}$ admits local times distributed as

$$(\sqrt{a}L^e_{a^{-3/2}t}(x))_{x\in\mathcal{T},t\geq 0}$$

that, $\mathbf{P}_\varrho^{\Theta_a(e)}$-a.s., are jointly continuous in (x,t). To justify this, it takes two steps to check that they satisfy, $\mathbf{P}_\varrho^{\Theta_a(e)}$-a.s., the occupation density formula (see [38, Lemma 2.4] and the references that lie in the proof of (b) and (2.6)):

$$\begin{aligned}\int_{[x,y]} \sqrt{a}L^e_{a^{-3/2}u}(z)\mu_{\mathcal{T}_{\Theta_a(e)}}(dz) &= \int_{[\tilde{x}_a,\tilde{y}_a]} a^{3/2}L^e_{a^{-3/2}u}(z)\mu_{\mathcal{T}_e}(dz) \\ &= \int_0^{a^{-3/2}u} a^{3/2}\mathbb{1}_{[\tilde{x}_a,\tilde{y}_a]}(X_k^e)dk \\ &= \int_0^u \mathbb{1}_{[\tilde{x}_a,\tilde{y}_a]}(X^e_{a^{-3/2}k})dk,\end{aligned}$$

for every $t \geq 0$, where the first equality is obtained by (4.11) and the second holds, $\mathbf{P}_\varrho^e$-a.s., by the occupation density formula applied to $(L_t^e(x))_{x\in\mathcal{T},t\geq 0}$. In addition, for $a > 1$,

$$\{X^e_{a^{-3/2}t}, \mathbf{P}_\varrho^e(;[\tilde{x}_a,\tilde{y}_a])\} \stackrel{(d)}{=} \{X_t^{\Theta_a(e)}, \mathbf{P}_\varrho^{\Theta_a(e)}(;[x,y])\},$$

where $\stackrel{(d)}{=}$ means equality in distribution (to justify why the processes are equal in law, see the definition of a speed motion on a compact real tree after Theorem

5.1.1),which brings us to our second step, confirming that, $\mathbf{P}_\varrho^{\Theta_a(e)}$-a.s.,

$$\int_{[x,y]} \sqrt{a} L^e_{a^{-3/2}u}(z)\mu_{\mathcal{T}_{\Theta_a(e)}}(dz) = \int_0^u \mathbb{1}_{[\tilde{x}_a,\tilde{y}_a]}(X^e_{a^{-3/2}k})dk = \int_0^u \mathbb{1}_{[x,y]}(X^{\Theta_a(e)}_k)dk,$$

for every open line segment $[x,y] \subseteq \mathcal{T}_{\Theta_a(e)}$ with $x,y \in \mathcal{T}'$, and $t \geq 0$. In view of (4.9), this can be seen to hold for any $A \cap \mathrm{Sk}(\mathcal{T}_{\Theta_a(e)})$ with $A \in \mathcal{B}(\mathcal{T}_{\Theta_a(e)})$.

Now, for every $\varepsilon \in (0,1)$ fraction of time and every scalar parameter $a > 1$, for the ε-blanket time variable of the Brownian motion on $\Theta_a\mathcal{T}$ as defined in (3.6), we have that

$$\begin{aligned}\tau_{\mathrm{bl}}^{\Theta_a(e)}(a^{-1}\varepsilon) &\overset{(d)}{=} \inf\{t \geq 0 : \sqrt{a}L_{a^{-3/2}t}(x) \geq \varepsilon a^{-1}t,\ \forall x \in \mathcal{T}_e\} \\ &\overset{(d)}{=} \inf\{t \geq 0 : L_{a^{-3/2}t}(x) \geq \varepsilon a^{-3/2}t,\ \forall x \in \mathcal{T}_e\} \overset{(d)}{=} a^{-3/2}\tau^e_{\mathrm{bl}}(\varepsilon).\end{aligned}$$

This implies that

$$\mathcal{T} \in \mathcal{A}_\varepsilon \text{ if and only if } \Theta_a\mathcal{T} \in \mathcal{A}_{a^{-1}\varepsilon}. \tag{4.12}$$

In other words, $\tau^e_{\mathrm{bl}}(\varepsilon)$ is continuous at ε, $\mathbf{P}^e_\varrho$-a.s., if and only if $\tau_{\mathrm{bl}}^{\Theta_a(e)}(a^{-1}\varepsilon)$ is continuous at ε, $\mathbf{P}^e_\varrho$-a.s. Using the way in which the blanket times above relate as well as the scaling properties of the usual and the normalized Itô excursion we prove the following proposition.

Proposition 4.1.5. *For every $\varepsilon \in (0,1)$, $\mathbb{N}$-a.e. $e \in E$, $\tau^e_{\mathrm{bl}}(\varepsilon)$ is continuous at ε, $\mathbf{P}^e_\varrho$-a.s. Moreover, $\mathbb{N}_1$-a.e. $e \in E$, $\tau^e_{\mathrm{bl}}(\varepsilon)$ is continuous at ε, $\mathbf{P}^e_\varrho$-a.s.*

Proof. Fix $\varepsilon \in (0,1)$. We choose $a > 1$ in such a way that $a^{-1}\varepsilon \in \Omega_0$, where Ω_0 is the set for which the assertion in (4.8) holds ℓ almost-everywhere. Namely, $\mathbb{N}$-a.e. $e \in E$, $\mathcal{T} \in \mathcal{A}_{a^{-1}\varepsilon}$. Using the scaling property of Itô's excursion measure as quoted in (2.5) yields, $\sqrt{a}\mathbb{N}$-a.e. $e \in E$, $\Theta_a\mathcal{T} \in \mathcal{A}_{a^{-1}\varepsilon}$, and consequently $\mathbb{N}$-a.e. $e \in E$, $\mathcal{T} \in \mathcal{A}_\varepsilon$, where we exploited (4.12). Since ε was arbitrary, this establishes our first conclusion.

What remains now is to prove a similar result but with $\mathbb{N}(de)$ replaced with its version conditioned on the length of the excursion. Following the same steps we used in order to prove (4.8), we infer that for ℓ-a.e. $\varepsilon \in (0,1)$, $\mathbb{N}(\cdot|\zeta \in [1,2])$-a.e. $e \in E$, $\mathcal{T} \in \mathcal{A}_{\zeta^{-1}\varepsilon}$, and consequently ℓ-a.e. $\varepsilon \in (0,1)$, $\mathbb{N}(\cdot|\zeta \in [1,2])$-a.e. $e \in E$, $\Theta_\zeta\mathcal{T} \in \mathcal{A}_\varepsilon$. Using the scaling property of the normalized Itô excursion measure quoted in (2.7), we deduce that ℓ-a.e. $\varepsilon \in (0,1)$, $\mathbb{N}_1$-a.e. $e \in E$, $\mathcal{T} \in \mathcal{A}_\varepsilon$, where $\mathbb{N}_1$ is the law of the normalized Brownian excursion. To conclude, we proceed using the same argument as in the first paragraph of the proof. Fix an $\varepsilon \in (0,1)$ and choose $a > 1$ in such a way that $a^{-1}\varepsilon \in \Phi_0$, where Φ_0 is the set for which the assertion $\mathbb{N}_1$-a.e. $e \in E$, $\mathcal{T} \in \mathcal{A}_\varepsilon$ holds ℓ almost-equally. Namely, $\mathbb{N}_1$-a.e. $e \in E$, $\mathcal{T} \in \mathcal{A}_{a^{-1}\varepsilon}$, which from the scaling property of the normalized Itô excursion measure yields $a\mathbb{N}_1$-a.e. $e \in E$, $\Theta_a \mathcal{T} \in \mathcal{A}_{a^{-1}\varepsilon}$. As before this gives us that $\mathbb{N}_1$-a.e. $e \in E$, $\mathcal{T} \in \mathcal{A}_\varepsilon$, or in other words that $\mathbb{N}_1$-a.e. $e \in E$, $\tau_{\text{bl}}^e(\varepsilon)$ is continuous at ε, $\mathbf{P}_\varrho^e$-a.s.

□

Proof of Theorem 4.0.3. Since the space in the convergence in (4.2) is separable, we can use Skorohod's coupling to deduce that there exists a common metric space (F, d_F) and a joint probability measure $\tilde{\mathbf{P}}$ such that, as $n \to \infty$,

$$d_H^F(V(\tilde{\mathcal{T}}_n), \tilde{\mathcal{T}}_e) \to 0, \qquad d_P^F(\tilde{\pi}^n, \tilde{\mu}_{\mathcal{T}_e}) \to 0, \qquad d_F(\tilde{\varrho}^n, \tilde{\varrho}) \to 0, \qquad \tilde{\mathbf{P}}\text{ -a.s.},$$

where

$$((V(\mathcal{T}_n), d_{\mathcal{T}_n}, \varrho^n), \pi^n) \overset{(d)}{=} ((V(\tilde{\mathcal{T}}_n), d_{\tilde{\mathcal{T}}_n}, \tilde{\varrho}^n), \tilde{\pi}^n)$$

and

$$((\mathcal{T}_e, \mu_{\mathcal{T}_e}, \varrho), \mu_{\mathcal{T}_e}) \overset{(d)}{=} ((\tilde{\mathcal{T}}_e, \mu_{\tilde{\mathcal{T}}_e}, \tilde{\varrho}), \tilde{\mu}_{\mathcal{T}_e}).$$

Moreover, X^n under $\mathbf{P}_{\tilde{\varrho}^n}^{\tilde{\mathcal{T}}_n}$ converges weakly to X^e under $\mathbf{P}_{\tilde{\varrho}}^{\tilde{e}}$ on $D([0,1], F)$. In Proposition 4.1.4, we proved equicontinuity of the local times with respect to the annealed law. Reexamining the proof of Lemma 3.2.2, one can see that in this case L^n under $\mathbb{P}_{\tilde{\varrho}^n}(\cdot) := \int \mathbf{P}_{\tilde{\varrho}^n}^{\tilde{\mathcal{T}}_n}(\cdot) d\tilde{\mathbf{P}}$ will converge weakly to L under $\mathbb{P}_{\tilde{\varrho}}(\cdot) := \int \mathbf{P}_{\tilde{\varrho}}^{\tilde{e}}(\cdot) d\tilde{\mathbf{P}}$ in the sense of the local convergence as stated in (3.11). It was this precise statement that was used extensively in the derivation of asymptotic distributional bounds for the blanket times in Section 3.3. Then, the statement of Theorem 3.1.2 translates as follows. For every $\varepsilon \in (0,1)$, $\delta \in (0,1)$ and $t \in [0,1]$,

$$\limsup_{n\to\infty} \int \mathbf{P}_{\tilde{\varrho}^n}^{\tilde{\mathcal{T}}_n} \left(n^{-3/2}\tau_{\text{bl}}^n(\varepsilon) \le t\right) d\tilde{\mathbf{P}} \le \int \mathbf{P}_{\tilde{\varrho}}^{\tilde{e}} \left(\tau_{\text{bl}}^e(\varepsilon(1-\delta)) \le t\right) d\tilde{\mathbf{P}},$$

$$\liminf_{n\to\infty} \int \mathbf{P}_{\tilde{\varrho}^n}^{\tilde{\mathcal{T}}_n} \left(n^{-3/2}\tau_{\text{bl}}^n(\varepsilon) \le t\right) d\tilde{\mathbf{P}} \ge \int \mathbf{P}_{\tilde{\varrho}}^{\tilde{e}} \left(\tau_{\text{bl}}^e(\varepsilon(1+\delta)) < t\right) d\tilde{\mathbf{P}}.$$

From Proposition 4.1.5 and the dominated convergence theorem, we have that for every $\varepsilon \in (0,1)$ and $t \in [0,1]$,

$$\lim_{\delta\to 0}\int \mathbf{P}^{\tilde{e}}_{\tilde{\varrho}}\left(\tau^{e}_{\mathrm{bl}}(\varepsilon(1\pm\delta)) \le t\right) d\tilde{\mathbf{P}} = \lim_{\delta\to 0}\int \mathbf{P}^{\tilde{e}}_{\tilde{\varrho}}\left(\tau^{e}_{\mathrm{bl}}(\varepsilon) < t\right) d\tilde{\mathbf{P}}$$
$$= \int \mathbf{P}^{e}_{\varrho}\left(\tau^{e}_{\mathrm{bl}}(\varepsilon) \le t\right) \mathbb{N}(de) = \mathbb{P}_{\varrho}\left(\tau^{e}_{\mathrm{bl}}(\varepsilon) \le t\right).$$

Therefore, we deduce that for every $\varepsilon \in (0,1)$ and $t \in [0,1]$,

$$\lim_{n\to\infty}\mathbb{P}_{\varrho^n}\left(n^{-3/2}\tau^{n}_{\mathrm{bl}}(\varepsilon) \le t\right) = \lim_{n\to\infty}\int \mathbf{P}^{\mathcal{T}_n}_{\varrho^n}\left(n^{-3/2}\tau^{n}_{\mathrm{bl}}(\varepsilon) \le t\right) P(d\mathcal{T}_n)$$
$$= \mathbb{P}_{\varrho}\left(\tau^{e}_{\mathrm{bl}}(\varepsilon) \le t\right).$$

□

4.2 The critical Erdős-Rényi random graph

Our interest in this section shifts to the Erdős-Rényi random graph at criticality. Take n vertices labeled by $[n] = \{1, ..., n\}$ and put edges between any pair independently with fixed probability $p \in [0,1]$. Denote the resulting random graph by $G(n,p)$. Let $p = c/n$, for some $c > 0$. This model exhibits a phase transition in its structure for large n, as it was discovered in the groundbreaking work of Erdős and Rényi in [51]. With probability tending to 1, when $c < 1$, the largest connected component has size $O(\log n)$. On the other hand, when $c > 1$, we see the emergence of a giant component that contains a positive proportion of the vertices. In the critical case, when $c = 1$, they showed that the largest components of $G(n,p)$ have size of order $n^{2/3}$.

We will focus here on the critical case $c = 1$, and more specifically, in the critical window $p = n^{-1} + \lambda n^{-4/3}$, $\lambda \in \mathbb{R}$. The most significant result in this regime was proven by Aldous [8]. Fix $\lambda \in \mathbb{R}$ and let $(C_i^n)_{i\ge 1}$ denote the sequence of the component sizes of $G(n, n^{-1} + \lambda n^{-4/3})$. For reasons that are inherent in understanding the structure of the components, we track the surplus of each one, that is the number of vertices that have to be removed in order to obtain a tree. Let $(S_i^n)_{n\ge 1}$ be the sequence of the corresponding surpluses.

Theorem 4.2.1 (Aldous [8]). *As $n \to \infty$,*

$$\left(n^{-2/3}(C_i^n)_{i\ge 1}, (S_i^n)_{i\ge 1}\right) \longrightarrow ((C_i)_{i\ge 1}, (S_i)_{i\ge 1})$$

in distribution, where the convergence of the first sequence takes place in $\ell^2_\downarrow$*, the set of positive, decreasing sequences* $(x_i)_{i\geq 1}$ *with* $\sum_{i=1}^{\infty} x_i^2 < \infty$*. For the second sequence, the convergence takes place in the product topology.*

The limit is described by stochastic processes that encode various aspects of the structure of the random graph. Consider a Brownian motion with parabolic drift:

$$B_t^\lambda := B_t + \lambda t - \frac{t^2}{2}, \qquad t \geq 0, \tag{4.13}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Then, the limiting sequence $(C_i)_{i\geq 1}$ has the distribution of the ordered sequence of lengths of excursions of the process

$$B_t^\lambda - \inf_{0\leq s\leq t} B_s^\lambda, \qquad t \geq 0,$$

that is the parabolic Brownian motion reflected upon its minimum. Finally, $(S_i)_{i\geq 1}$ is recovered as follows. Draw the graph of the reflected process and scatter points on the plane according to a rate 1 Poisson process and keep those that fall between the x-axis and the function. Then, S_i are the Poisson number of points that fell in the corresponding excursion with length C_i. Observe that the distribution of the limit $(C_i)_{i\geq 1}$ depends on the particular value of λ chosen.

The scaling limit of the largest connected component of the Erdős-Rényi random graph in the critical window arises as a tilted version of the Brownian CRT following a procedure introduced in [3]. Given a pointset $\mathcal{P}$, that is a subset of the upper half plane that contains only a finite number of points in any compact subset, and a positive excursion e, we define $\mathcal{P} \cap e$ as the number of points from $\mathcal{P}$ that fall under the area of e. We construct a "glued" metric space $\mathcal{M}_{e,\mathcal{P}}$ as follows. For each point $(t,x) \in \mathcal{P} \cap e$, let $u_{(t,x)}$ be the unique vertex $p_e(t) \in \mathcal{T}_e$ and $v_{(t,x)}$ be the unique vertex on the path from the root to $u_{(t,x)}$ at a distance x from the root. Let $E_\mathcal{P} = \{(u_{(t,x)}, v_{(t,x)}) : (t,x) \in \mathcal{P} \cap e\}$ be the finite set that consists of the pairs of vertices to be identified. Let $\{v_i, u_i\}_{i=1,\ldots,k}$ be k pairs of points that belong to $E_\mathcal{P}$. We define a quasi-metric on $\mathcal{T}_e$ by setting:

$$d_{\mathcal{M}_{e,\mathcal{P}}}(x,y) := \min\left\{ d_e(x,y), \inf_{i_1,\ldots,i_r} \left\{ d_e(x,u_{i_1}) + \sum_{j=1}^{r-1} d_e(v_{i_j}, u_{i_{j+1}}) + d_e(v_r, y) \right\}\right\}, \tag{4.14}$$

where the infimum is taken over r positive integers, and all subsets $\{i_1, ..., i_r\} \subseteq \{1, ..., k\}$. Moreover, note that the vertices $i_1, ..., i_k$ can be chosen to be distinct.

The metric defined above gives the shortest distance between $x, y \in \mathcal{T}_e$ when we glue the vertices v_i and u_i together for $i = 1, ..., k$. It is clear that $d_{\mathcal{M}_{e,\mathcal{P}}}$ defines only a quasi-metric since $d_{\mathcal{M}_{e,\mathcal{P}}}(u_i, v_i) = 0$, for every $i = 1, ..., k$, but $u_i \neq v_i$, for every $i = 1, ..., k$. We define an equivalence relation on $\mathcal{T}_e$ by setting $x \sim_{E_{\mathcal{P}}} y$ if and only if $d_{\mathcal{M}_{e,\mathcal{P}}}(x, y) = 0$. This makes the vertex identification explicit and $\mathcal{M}_{e,\mathcal{P}}$ is defined as

$$\mathcal{M}_{e,\mathcal{P}} := (\mathcal{T}_e / \sim_{E_{\mathcal{P}}}, d_{\mathcal{M}_{e,\mathcal{P}}}).$$

To endow $\mathcal{M}_{e,\mathcal{P}}$ with a canonical measure, let $p_{e,\mathcal{P}}$ denote the canonical projection from $\mathcal{T}_e$ to the quotient space $\mathcal{T}_e / \sim_{E_{\mathcal{P}}}$. We define $\pi_{e,\mathcal{P}} := \mu_{\mathcal{T}_e} \circ p_{e,\mathcal{P}}^{-1}$, where $\mu_{\mathcal{T}_e}$ is the image measure on $\mathcal{T}_e$ of the Lebesgue measure ℓ on $[0, \zeta]$ by the canonical projection p_e of $[0, \zeta]$ onto $\mathcal{T}_e$. So, $\pi_{e,\mathcal{P}} = (\ell \circ p_e^{-1}) \circ p_{e,\mathcal{P}}^{-1}$. We note that the restriction of $p_{e,\mathcal{P}}$ to $\mathcal{T}_e$ is p_e.

For every $\zeta > 0$, as in [3], we define a tilted excursion of length ζ to be a random variable that takes values in E, whose distribution is characterized by

$$\mathbf{P}(\tilde{e} \in \mathcal{E}) = \frac{\mathbf{E}\left(\mathbb{1}_{\{e \in \mathcal{E}\}} \exp\left(\int_0^\zeta e(t)dt\right)\right)}{\mathbf{E}\left(\exp\left(\int_0^\zeta e(t)dt\right)\right)},$$

for every measurable $\mathcal{E} \subseteq E$. We note here that the σ-algebra on E is the one generated by the open sets with respect to the supremum norm on $C(\mathbb{R}_+, \mathbb{R}_+)$. Write $\mathcal{M}^{(\zeta)}$ for the random compact metric space distributed as $(\mathcal{M}_{\tilde{e},\mathcal{P}}, 2d_{\mathcal{M}_{\tilde{e},\mathcal{P}}})$, where $\tilde{e}$ is a tilted Brownian excursion of length ζ and the random pointset of interest $\mathcal{P}$ is a Poisson point process on $\mathbb{R}_+^2$ of unit intensity with respect to the Lebesgue measure, independent of $\tilde{e}$.

We now give an alternative description of $\mathcal{M}_{\tilde{e},\mathcal{P}}$, for which the full details can be found in [3, Proposition 20]. From the construction, it is easy to prove that the number $|\mathcal{P} \cap \tilde{e}|$ of vertex identifications is a Poisson random variable with mean $\int_0^\zeta \tilde{e}(u)du$. Given $|\mathcal{P} \cap \tilde{e}| = k$, the co-ordinate $u_{(t,x)}$ has density

$$\frac{\tilde{e}(u)}{\int_0^\zeta \tilde{e}(t)dt}$$

on $[0, \zeta]$, and given $u_{(t,x)}$, its pair $v_{(t,x)}$ is uniformly distributed on $[0, \tilde{e}(u_{(t,x)})]$. The other $k-1$ vertex identifications are distributed accordingly and independently of $(u_{(t,x)}, v_{(t,x)})$.

After introducing notation, we are in the position to write the limit of the largest connected component, say $\mathcal{C}_1^n$, as $\mathcal{M}^{(C_1)}$, where C_1 has the distribution of the length of the longest excursion of the reflected upon its minimum parabolic Brownian motion in (4.13). Moreover, the longest excursion, when conditioned to have length C_1, is distributed as a tilted excursion $\tilde{e}$ with length C_1. The following convergence is a simplified version of [3, Theorem 2]. As $n \to \infty$,

$$\left(n^{-2/3} C_1^n, \left(V(\mathcal{C}_1^n), n^{-1/3} d_{\mathcal{C}_1^n}\right)\right) \longrightarrow \left(C_1, (\mathcal{M}, d_{\mathcal{M}})\right), \tag{4.15}$$

in distribution, where conditional on C_1, $\mathcal{M} \stackrel{(d)}{=} \mathcal{M}^{(C_1)}$. Moreover, it was shown in [34], that the discrete-time simple random walk $X^{\mathcal{C}_1^n}$ on $\mathcal{C}_1^n$, started from a distinguished vertex ϱ^n, satisfies a distributional convergence of the form

$$\left(n^{-1/3} X^{\mathcal{C}_1^n}_{\lfloor nt \rfloor}\right)_{t \geq 0} \to \left(X_t^{\mathcal{M}}\right)_{t \geq 0}, \tag{4.16}$$

where $X^{\mathcal{M}}$ is a diffusion on $\mathcal{M}$, started from a distinguished point $\varrho \in \mathcal{M}$. The convergence of the associated stationary probability measures, say π^n, was not directly proven in [34], although the hard work required has been done. More specifically, see [34, Lemma 6.3]. The results above can be reformulated in the following distributional convergence in terms of the extended pointed Gromov-Hausdorff topology:

$$\left(\left(V(\mathcal{C}_1^n), n^{-1/3} d_{\mathcal{C}_1^n}, \varrho^n\right), \pi^n, \left(n^{-1/3} X^{\mathcal{C}_1^n}_{\lfloor nt \rfloor}\right)_{t \geq 0}\right) \longrightarrow \left((\mathcal{M}, d_{\mathcal{M}}, \varrho), \pi^{\mathcal{M}}, X^{\mathcal{M}}\right). \tag{4.17}$$

Now, we describe how to generate a connected component on a fixed number of vertices, for which the full details can be found in [3, Lemma 6] and [3, Lemma 7]. To any such component we can associate a spanning subtree, the depth-first tree by considering the following algorithm. The initial step places the vertex with label 1 in a stack and declares it open. In the next step vertex 1 is declared as explored and is removed from the top of the stack, where we place in increasing order the neighbors of 1 that have not been seen (open or explored) yet, while declaring them open. We proceed inductively. When the set of open vertices becomes empty the algorithm terminates. It is obvious that the resulting graph that consists of edges between a vertex that was explored at a given step and a vertex that has not been seen yet at the same step, is a tree. For a connected graph G with m vertices, we refer to this tree as the depth-first tree and write $T(G)$. For $i = 0, ..., m-1$, let $X(i)$ be the number of vertices seen but not yet fully explored at step i. The process $(X(i) : 0 \leq i < m)$ is called the depth-first walk of G.

Let $\mathbb{T}_m$ be the set of (unordered) trees labeled by $[m]$. For $T \in \mathbb{T}_m$, its associated depth-first tree is T itself. We call an edge permitted by the depth-first procedure run on T if its addition produces the same depth-first tree. Exactly $X(i)$ edges are permitted at step i, and therefore the total number of permitted edges is given by

$$a(T) := \sum_{i=0}^{m-1} X(i),$$

which is called the area of T. Given a tree T and a connected graph G, $T(G) = T$ if and only if G can be obtained from T by adding a subset of permitted edges by the depth-first procedure. Therefore, writing $\mathbb{G}_T$ for the set of connected graphs G that satisfy $T(G) = T$, we have that $\{\mathbb{G}_T : T \in \mathbb{T}_m\}$ is a partition of the connected graphs on $[m]$, and that the cardinality of $\mathbb{G}_T$ is $2^{a(T)}$, since every permitted edge is included or not.

Back to the question on how to generate a connected component, write G_m^p for the graph with the same distribution as $G(m, p)$ conditioned to be connected. We focus on generating G_m^p instead.

Lemma 4.2.2 (**Addario-Berry, Broutin, Goldschmidt** [3]). *Fix $p \in (0, 1)$. Pick a random tree $\tilde{T}_m^p$ that has a "tilted" distribution which is biased in favor of trees with large area. Namely, pick $\tilde{T}_m^p$ in such a way that*

$$P(\tilde{T}_m^p = T) \propto (1-p)^{-a(T)}, \qquad T \in \mathbb{T}_m.$$

Add to $\tilde{T}_m^p$ each of the $a(\tilde{T}_m^p)$ permitted edges independently with probability p. Call the graph generated $\tilde{G}_m^p$. Then, $\tilde{G}_m^p$ has the same distribution as G_m^p.

Proof. For a connected graph G on $[m]$, let $s(G) := |E(G)| - (m-1)$ denote its surplus. From the definition of G_m^p, for a connected graph G on $[m]$, we have that

$$\begin{aligned} P(G_m^p = G) \propto p^{|E(G)|}(1-p)^{\binom{m}{2}-|E(G)|} &= p^{s(G)+m-1}(1-p)^{\binom{m}{2}-s(G)-m+1} \\ &\propto p^{s(G)}(1-p)^{-s(G)}. \end{aligned}$$

Also, note that $T(\tilde{G}^p_m) = \tilde{T}^p_m$, and therefore

$$\begin{aligned} P(\tilde{G}^p_m = G) &\propto (1-p)^{-a(T)} P(\tilde{G}^p_m = G | T(G) = T) \\ &= (1-p)^{-a(T)} p^{s(G)} (1-p)^{a(T)-s(G)} = p^{s(G)}(1-p)^{-s(G)}, \end{aligned}$$

which completes the proof.

□

We use ϱ^m to denote the root of $\tilde{T}^p_m$. In what follows we give a detailed description of how we can transfer the results proved in Section 4.1. We denote by $\tilde{C}_m := (\tilde{C}_m(i) : 0 \le i \le 2m)$ the contour function of $\tilde{T}^p_m$, and by

$$\tilde{C}_{(m)}(s) := \frac{\tilde{C}_m(2(m/\zeta)s)}{\sqrt{(m/\zeta)}}, \qquad 0 \le s \le \zeta,$$

its normalized contour function of positive length ζ. We start by showing that, for some $\alpha > 0$, the sequence $||\tilde{C}_{(m)}||_{H_\alpha}$ of Hölder norms is tight.

Lemma 4.2.3. *Suppose that p=p(m) is in such a way that $mp^{2/3} \to \zeta$, as $m \to \infty$. There exists $\alpha > 0$, such that*

$$\lim_{M\to\infty} \liminf_{m\to\infty} P\left(\sup_{0\le s\ne t\le 1} \frac{|\tilde{C}_{(m)}(s) - \tilde{C}_{(m)}(t)|}{|t-s|^\alpha} \le M \right) = 1.$$

Proof. We simply assume that $\zeta = 1$. The general result follows by Brownian scaling. Let T_m be a tree chosen uniformly from $[m]$. We note here that Theorem 4.1.3 is stated in the more general framework of size-conditioned Galton-Watson trees with critical offspring distribution that has finite variance and exponential moments. If the offspring is distributed according to a Poisson with mean 1, then the conditioned tree is a uniformly distributed labeled tree (e.g. [56, Proposition 2.3]). By Lemma 4.2.2,

$$\begin{aligned} &P\left(\sup_{0\le s\ne t\le 1} \frac{|\tilde{C}_{(m)}(s) - \tilde{C}_{(m)}(t)|}{|t-s|^\alpha} \ge M \right) \\ &= \frac{E\left[\mathbb{1}_{\left\{ \sup_{0\le s\ne t\le 1} \frac{|C_{(m)}(s) - C_{(m)}(t)|}{|t-s|^\alpha} \ge M \right\}} (1-p)^{-a(T_m)} \right]}{E\left[(1-p)^{-a(T_m)}\right]}. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we have that

$$P\left(\sup_{0\leq s\neq t\leq 1}\frac{|\tilde{C}_{(m)}(s)-\tilde{C}_{(m)}(t)|}{|t-s|^{\alpha}}\geq M\right)$$

$$\leq \frac{P\left(\sup_{0\leq s\neq t\leq 1}\frac{|C_{(m)}(s)-C_{(m)}(t)|}{|t-s|^{\alpha}}\geq M\right)^{1/2}\left(E\left[(1-p)^{-2a(T_m)}\right]\right)^{1/2}}{E\left[(1-p)^{-a(T_m)}\right]}. \tag{4.18}$$

Since $mp^{2/3}\to 1$, as $m\to\infty$, there exists $c>0$ such that $p\leq cm^{-3/2}$, for every $m\geq 1$. Since T_m is a uniform random tree on $[m]$, from [3, Lemma 14], we can find universal constants K_1, $K_2>0$, such that

$$E\left[(1-p)^{-\xi a(T_m)}\right] < K_1 e^{K_2 c^2 \xi^2}, \tag{4.19}$$

for fixed $\xi>0$. Recall that $a(T_m)=\sum_{i=0}^{m-1}X_m(i)$, where $(X_m(i):0\leq i\leq m)$ is the depth-first walk associated with T_m (for convenience, we have put $X_m(m)=0$). From [84, Theorem 3], we know that,

$$(m^{-1/2}X_m(\lfloor mt\rfloor))_{t\in[0,1]}\to(e(t))_{t\in[0,1]},$$

as $m\to\infty$, in distribution in $D([0,1],\mathbb{R}_+)$, where $(e(t))_{t\in[0,1]}$ is a normalized Brownian excursion. Writing

$$(1-p)^{-a(T_m)}=(1-p)^{-\sum_{i=0}^{m-1}X_m(i)}=(1-p)^{-m^{3/2}\int_0^1 m^{-1/2}X_m(\lfloor mt\rfloor)dt},$$

and using that the sequence $(1-p)^{-a(T_m)}$ is uniformly integrable, see (4.19), we deduce that

$$E\left[(1-p)^{-a(T_m)}\right]\to E\left[\exp\left(\int_0^1 e(u)du\right)\right]>0, \tag{4.20}$$

as $m\to\infty$. Thus, for m large enough,

$$\left(E\left[(1-p)^{-2a(T_m)}\right]\right)^{1/2}/E[(1-p)^{-a(T_m)}]$$

is bounded by a universal constant, see (4.19) and (4.20). To conclude, taking first $m\to\infty$ and then $M\to\infty$, the desired result follows from (4.18) and Theorem 4.1.3. $\square$

It is now immediate to check that the local times $(L_t^m(x))_{x\in V(G_m^p),t\geq 0}$ of the corresponding simple random walk on G_m^p are equicontinuous under the annealed law. The proof of the next lemma relies heavily on the same methods used to establish Proposition 4.1.4, and therefore we will make use of the parts that remain unchanged.

Recall that the graph generated by the process of adding $\mathrm{Bin}(a(\tilde{T}_m^p),p)$ number of surplus edges to $\tilde{T}_m^p$ was denoted by $\tilde{G}_m^p$. We view $\tilde{G}_m^p$ as the metric space $\tilde{T}_m^p$ that includes the edges (of length 1) that have been added and we equip it with the resistance metric $R_{\tilde{G}_m^p}$ defined by (3.3).

Lemma 4.2.4. *Suppose that p=p(m) is such that* $mp^{2/3}\to\zeta$, *as* $m\to\infty$. *For every* $\varepsilon>0$ *and* $T>0$,

$$\lim_{\delta\to 0}\limsup_{m\to\infty}\mathbb{P}_{\varrho^m}\left(\sup_{\substack{y,z\in V(G_m^p):\\ m^{-1/2}R_{G_m^p}(y,z)<\delta}}\sup_{t\in[0,T]}m^{-1/2}|L^m_{m^{3/2}t}(y)-L^m_{m^{3/2}t}(z)|\geq\varepsilon\,\middle|\,s(G_m^p)=s\right)$$
$$=0.$$

Proof. We simply assume that $\zeta=1$. The general result follows by Brownian scaling. From Lemma 4.2.3, given $t_1,t_2\in[0,1]$, with $2nt_1$ and $2nt_2$ integers, such that $p_{\tilde{C}_{(m)}}(t_1)=y$ and $p_{\tilde{C}_{(m)}}(t_2)=z$, there exist $M>0$ and $\alpha>0$, such that

$$d_{\tilde{C}_{(m)}}(t_1,t_2)=\tilde{C}_{(m)}(t_1)+\tilde{C}_{(m)}(t_2)-2\min_{r\in[t_1\wedge t_2,t_1\vee t_2]}\tilde{C}_{(m)}\leq 2M|t_1-t_2|^{\alpha},\tag{4.21}$$

with probability arbitrarily close to 1, cf. (4.4). Conditioned on $\tilde{C}_{(m)}$ satisfying (4.21), the resistance between y and z on $\tilde{G}_m^p$ is smaller than the total length of the path between y and z on $\tilde{T}_m^p$, see Proposition A.0.1. Therefore,

$$R_{\tilde{G}_m^p}(y,z)\leq d_{\tilde{T}_m^p}(y,z)=m^{1/2}d_{\tilde{v}_m}(t_1,t_2)\leq 2Mm^{1/2}|t_1-t_2|^{\alpha},$$

which indicates that, on the event that (4.21) holds, the maximum resistance of $\tilde{G}_m^p$ is bounded above by a multiple of $m^{1/2}$. More specifically,

$$r(\tilde{G}_m^p)\leq Mm^{1/2}2^{\alpha+1}.$$

Moreover, $m(\tilde{G}_m^p)=2E(\tilde{G}_m^p)=2(s(\tilde{G}_m^p)+m-1)$. An application of Theorem

4.1.2, which was originally formulated for the local times of random walks on weighted graphs in terms of the resistance metric, yields

$$\mathbf{E}_{\varrho^m}^{\tilde{G}_m^p}\left[\left|\left|m^{-1/2}\left(L^m_{m^{3/2}\cdot}(y)-L^m_{m^{3/2}\cdot}(z)\right)\right|\right|^p_{\infty,[0,T]}\left|s(\tilde{G}_m^p)=s\right.\right]\leq\tilde{c}_5\left|t_1-t_2\right|^{\alpha p/2},$$

conditional on $\tilde{C}_{(m)}$ satisfying (4.21), for any fixed $p \geq 2$, cf. (4.5). Since the discrete local time process is interpolated linearly between the integer time points $2nt_1$ and $2nt_2$, the statement above is also valid for every $t_1, t_2 \in [0,1]$. The rest of the proof is finished in the manner of Proposition 4.1.4, and therefore we omit it.

□

For notational simplicity, the next result is stated for the largest connected component $\mathcal{C}_1^n$ of $G(n, n^{-1}+\lambda n^{-4/3})$, for fixed $\lambda \in \mathbb{R}$. In fact, it holds for the i-th largest connected component. As usual, we denote by $(L_t^n(x))_{x\in V(\mathcal{C}_1^n), t\geq 0}$ the local times of the simple random walk on $\mathcal{C}_1^n$.

Proposition 4.2.5. *For every $\varepsilon > 0$ and $T > 0$,*

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}_{\varrho^n}\left(\sup_{\substack{y,z\in V(\mathcal{C}_1^n):\\ n^{-1/3}R_{\mathcal{C}_1^n}(y,z)<\delta}}\ \sup_{t\in[0,T]} n^{-1/3}|L^n_{nt}(y)-L^n_{nt}(z)|\geq\varepsilon\right)=0.$$

Proof. Fix $\varepsilon > 0$, $\delta > 0$ and $T > 0$. In the random graph $G(n,p)$, conditional on C_1^n,

$$\mathcal{C}_1^n \overset{(d)}{=} G^p_{C_1^n},$$

where as above $p = n^{-1}+\lambda n^{-4/3}$, for fixed $\lambda \in \mathbb{R}$. Note that $np \to 1$, as $n \to \infty$. By (4.15) and Skorohod's representation theorem, there exists a probability space and random variables $\tilde{C}_1^n$, $\tilde{\mathcal{C}}_1^n$, $n \geq 1$ and $\tilde{C}_1$, $\tilde{\mathcal{M}}$ defined on that space, such that $(\tilde{C}_1^n, \tilde{\mathcal{C}}_1^n) \overset{(d)}{=} (\tilde{C}_1, \tilde{\mathcal{M}})$ with $n^{-2/3}\tilde{C}_1^n \to \tilde{C}_1$, as $n \to \infty$, in the almost-sure sense. Conditioning on the size and surplus of $\mathcal{C}_1^n$, if we denote by B_n^δ the measurable event

$$B_n^\delta := \sup_{\substack{y,z\in V(\mathcal{C}_1^n):\\ n^{-1/3}R_{\mathcal{C}_1^n}(y,z)<\delta}}\ \sup_{t\in[0,T]} n^{-1/3}|L^n_{nt}(y)-L^n_{nt}(z)|\geq\varepsilon,$$

for large enough constants A (appears in (4.22)) and S (appears in (4.23)), note

that

$$\mathbb{P}_{\varrho^n}(B_n^\delta) \leq \int \mathbf{P}_{\varrho^n}^{\mathcal{C}_1^n}\left(B_n^\delta; A^{-1}n^{2/3} \leq C_1^n \leq An^{2/3}\right) P(d\mathcal{C}_1^n) \tag{4.22}$$

$$+ P(C_1^n \notin [A^{-1}n^{2/3}, An^{2/3}]).$$

Since $\tilde{C}_1^n$ and $p = p(n)$ are such that $\tilde{C}_1^n p^{2/3} \to \tilde{C}_1$, as $n \to \infty$, in the almost-sure sense, we can bound (4.22) above by

$$\mathbb{P}_{\varrho^n}\left(\sup_{\substack{y,z \in V(G^p_{C_1^n}): \\ (C_1^n)^{-1/2} R_{G^p_{C_1^n}}(y,z) < \delta'}} \sup_{t \in [0,T']} (C_1^n)^{-1/2} \left| L^n_{(C_1^n)^{3/2}t}(y) - L^n_{(C_1^n)^{3/2}t}(z) \right| \geq \varepsilon' \middle| s(G^p_{C_1^n}) \leq S\right)$$

$$+ P(C_1^n \notin [A^{-1}n^{2/3}, An^{2/3}]) + P(S_1^n > S), \tag{4.23}$$

for appropriate $\varepsilon' > 0$, $\delta' > 0$ and $T' > 0$ that only depend on ε, δ, T and A. By Theorem 4.2.1,

$$\lim_{A \to \infty} \limsup_{n \to \infty} P(C_1^n \notin [A^{-1}n^{2/3}, An^{2/3}]) = 0. \tag{4.24}$$

Furthermore, as $n \to \infty$,

$$S_1^n \xrightarrow{(d)} \text{Poi}\left(\int_0^\zeta \tilde{e}(u)du\right),$$

where $\text{Poi}\left(\int_0^\zeta \tilde{e}(t)dt\right)$ denotes a Poisson random variable with mean the area under a tilted excursion of length ζ, see [3, Corollary 23]. As a consequence, tightness of the process that encodes the surplus of $\mathcal{C}_1^n$ follows:

$$\lim_{S \to \infty} \limsup_{n \to \infty} P(S_1^n > S). \tag{4.25}$$

The proof is finished by combining (4.24) and (4.25) with the equicontinuity result of Lemma 4.2.4, see (4.23).

□

4.2.1 Continuity of blanket times of Brownian motion on $\mathcal{M}$

To prove continuity of the ε-blanket time of the Brownian motion on $\mathcal{M}$, we first define a σ-finite measure on the product space of positive excursions and random pointsets of $\mathbb{R}^2_+$. Throughout this section, for simplicity, we still use ℓ to denote the Lebesgue measure on $\mathbb{R}^2_+$. We define $\mathbf{N}(d(e, \mathcal{P}))$ by setting:

$$\begin{aligned}&\mathbf{N}(de, |\mathcal{P}| = k, (dx_1, ..., dx_k) \in B_1 \times ... \times B_k)\\ &:= \int_0^\infty f_L(l)\mathbb{N}_l(de)\frac{e^{-1}}{k!}\prod_{i=1}^{k}\frac{\ell(B_i \cap A_e)}{\ell(A_e)},\end{aligned} \tag{4.26}$$

where $f_L(l) := dl/\sqrt{2\pi l^3}$, $l \geq 0$ gives the density of the length of the excursion e, see (2.6), and $A_e := \{(t, x) : 0 \leq x \leq e(t)\}$ denotes the area under its graph. In other words, the measure picks first an excursion length according to $f_L(l)$ and, given $L = l$, it picks a Brownian excursion of that length. Then, independently of e, it chooses k points according to a Poisson with unit mean, which are distributed uniformly on the area under the graph of e.

It turns out that this is an easier measure to work with when applying our scaling argument to prove continuity of the blanket times. Also, as we will see later, $\mathbf{N}$ is absolutely continuous with respect to the canonical measure $\mathbf{N}^{t,\lambda}(d(e, \mathcal{P}))$ that first at time t picks a tilted Brownian excursion e of a randomly chosen length l, and then independently of e chooses k points distributed as a Poisson random variable with mean $\int_0^l e(t)dt$, which as before are distributed uniformly on the area under the graph of e. To fully describe this measure, let $\mathbb{N}^{t,\lambda}$ denote the measure (for excursions starting at time t) associated to $B_t^\lambda - \inf_{0\leq s\leq t} B_s^\lambda$, see (4.13), first stated by Aldous in [8]. We note that $\mathbb{N}^{t,\lambda} = \mathbb{N}^{0,\lambda-t}$ and thus it suffices to describe $\mathbb{N}^{0,\lambda}$, for every $\lambda \in \mathbb{R}$. For every measurable subset A,

$$\mathbb{N}^{0,\lambda}(A) = \int_0^\infty \mathbb{N}_l^{0,\lambda}(A)f_L(l)F_\lambda(l)\mathbb{N}_l\left(\exp\left(\int_0^l e(u)du\right)\right), \tag{4.27}$$

where $\mathbb{N}_l^{0,\lambda}$ is a shorthand for the excursion measure $\mathbb{N}^{0,\lambda}$, conditioned on the event $\{\tilde{L} = l\}$ and $F_\lambda(l) := \exp\left(-1/6\left(\lambda^3 + (l-\lambda)^3\right)\right)$. For simplicity, let

$$g_{\tilde{L}}(l, \lambda) := f_L(\lambda)F_\lambda(l)\mathbb{N}_l\left(\exp\left(\int_0^l e(u)du\right)\right).$$

In analogy with (4.26), we characterize $\mathbf{N}^{t,\lambda}(d(e,\mathcal{P}))$ by setting:

$$\mathbf{N}^{t,\lambda}(de, |\mathcal{P}| = k, (dx_1, ..., dx_k) \in A_1 \times ... \times A_k)$$
$$:= \int_0^\infty g_{\tilde{L}}(l, \lambda - t) \mathbb{N}_l^{t,\lambda}(de) \exp\left(-\int_0^l e(u)du\right) \frac{\left(\int_0^l e(u)du\right)^k}{k!} \prod_{i=1}^k \frac{\ell(A_i \cap A_e)}{\ell(A_e)}. \tag{4.28}$$

After calculations that involve the use of the Cameron-Martin-Girsanov formula [92, Chapter IX, (1.10) Theorem] (for the entirety of those calculations one can consult [3, Section 5]), one deduces that

$$\mathbb{N}_l^{t,\lambda}(de) = \exp\left(\int_0^l e(u)du\right) \frac{\mathbb{N}_l(de)}{\mathbb{N}_l\left(\exp\left(\int_0^l e(u)du\right)\right)},$$

and as a consequence the following expression for the Radon-Nikodym derivative is valid:

$$\frac{d\mathbf{N}^{t,\lambda}}{d\mathbf{N}} = \frac{F_{\lambda-t}(l)\left(\int_0^l e(u)du\right)^k/k!}{e^{-1}/k!}$$
$$= \exp\left(1 - \frac{1}{6}\left(\lambda^3 + (l - \lambda + t)^3\right)\right)\left(\int_0^l e(u)du\right)^k. \tag{4.29}$$

Recall that for every $b > 1$, the mapping $\Theta_b : E \to E$ is defined by setting

$$\Theta_b(e)(t) := \sqrt{b}e(t/b), \qquad t \in [0, b\zeta],$$

for every $e \in E$. As we saw in Subsection 4.1.1, it acts on the real tree coded by e scaling its distance and measure appropriately, see (4.10) and (4.11). Recall the alternative description of the "glued" metric space $\mathcal{M}_{e,\mathcal{P}}$, where e is a positive Brownian excursion of length ζ and $\mathcal{P}$ is a Poisson point process on $\mathbb{R}_+^2$ of unit intensity with respect to the Lebesgue measure independent of e. The number $|\mathcal{P} \cap e|$ of vertex identifications is a Poisson random variable with mean $\int_0^\zeta e(u)du$. As a result, the number of vertex identifications $|\mathcal{P} \cap \Theta_b(e)|$ has law given by a Poisson distribution with mean

$$\int_0^{b\zeta} \sqrt{b}e(u/b)du = b^{3/2} \int_0^\zeta e(u)du.$$

Moreover, conditioned on $|\mathcal{P} \cap e|$ and e, the coordinates of a point $(u_{(t,x)}, v_{(t,x)})$ in $\mathcal{P} \cap e$ have densities proportional to $e(u)$ for $u_{(t,x)}$ and, conditioned on $u_{(t,x)}$, $v_{(t,x)}$ is uniformly distributed on $[0, e(u_{(t,x)})]$. Then, conditioned on $|\mathcal{P} \cap \Theta_b(e)|$, the coordinates of a point $(u^b_{(t,x)}, v^b_{(t,x)})$ in $\mathcal{P} \cap \Theta_b(e)$ are equal in law to $bu_{(t,x)}$ in the case of $u^b_{(t,x)}$, and conditioned on $u^b_{(t,x)}$, $v^b_{(t,x)}$ is uniformly distributed on $[0, \sqrt{b}e(u_{(t,x)})]$. From now on, we use $\Theta_b(e, \mathcal{P})$ to denote the mapping from the product space of positive Brownian excursions and random pointsets of the upper half plane onto itself that applies $\Theta_b(e)$ to e and repositions the collection of points in $\mathcal{P}$ as described above.

From the definition of the quasi-metric $d_{\mathcal{M}_{e,\mathcal{P}}}$ in (4.14), we have that under the application of Θ_b, it rescales like $\sqrt{b}d_{\mathcal{M}_{e,\mathcal{P}}}$, a statement that should be understood in accordance with (4.10). Let $\mathcal{L}(\mathcal{T}_e)$ denote the leaves of $\mathcal{T}_e$, that is the set of points $\sigma \in \mathcal{T}_e$, such that $\mathcal{T}_e \setminus \{\sigma\}$ is connected, i.e. the complement of the set of leaves is the skeleton of $\mathcal{T}_e$. In particular, $\mathcal{L}(\mathcal{T}_e)$ is uncountable, and $\mu_{\mathcal{T}_e}(\mathcal{L}(\mathcal{T}_e)) = \zeta$. Consider the set

$$I = \{\sigma \in \mathcal{L}(\mathcal{T}_e) : p_{e,\mathcal{P}}(\sigma) \in B\},$$

for a measurable subset B of $\mathcal{M}_{e,\mathcal{P}}$, where $p_{e,\mathcal{P}}$ is the canonical projection from $\mathcal{T}_e$ to the resulting quotient space after the vertex identifications, made explicit by the equivalence relation $\sim_{E_{\mathcal{P}}}$. We endowed $\mathcal{M}_{e,\mathcal{P}}$ with the measure $\pi_{e,\mathcal{P}}$, that is the image measure on $\mathcal{M}_{e,\mathcal{P}}$ of $\mu_{\mathcal{T}_e}$ on $\mathcal{T}_e$ by the canonical projection $p_{e,\mathcal{P}}$ of $\mathcal{T}_e$ onto $\mathcal{M}_{e,\mathcal{P}}$. Then, by definition $\pi_{e,\mathcal{P}}(B) = \mu_{\mathcal{T}_e}(I)$, and consequently $\pi_{\Theta_b(e,\mathcal{P})}(B) = \mu_{\mathcal{T}_{\Theta_b(e)}}(I)$. As we examined before, under the application of Θ_b, $\mu_{\mathcal{T}_{\Theta_b(e)}}$ rescales like $b\mu_{\mathcal{T}_e}$, where once again this should be understood according to (4.11) and the notation that was introduced in the course of its derivation. Finally, since $\mathbb{N} \circ \Theta_b^{-1} = \sqrt{b}\mathbb{N}$, see (2.5), and using the fact that $\ell(A_i \cap A_e)/\ell(A_e)$ in (4.26) is scale invariant under Θ_b, we have that

$$\mathbf{N} \circ \Theta_b^{-1} = \sqrt{b}\mathbf{N}.$$

Therefore, considering $\mathbf{N}$ instead of $\mathbf{N}^{t,\lambda}$ is advantageous as it could easily be seen to enjoy the same scaling property as $\mathbb{N}$.

We now have all the ingredients to prove continuity of the blanket times of the Brownian motion on $\mathcal{M}$. We describe the arguments that have been already used in establishing Proposition 4.1.5. Let $\tau_{\mathrm{bl}}^{e,\mathcal{P}}(\varepsilon)$ denote the ε-blanket time of the Brownian motion $X^{e,\mathcal{P}}$ on $\mathcal{M}_{e,\mathcal{P}}$ started from a distinguished vertex $\bar{\varrho}$, for some $\varepsilon \in (0,1)$. Taking the expectation of the law of $\tau_{\mathrm{bl}}^{e,\mathcal{P}}(\varepsilon)$, $\varepsilon \in (0,1)$ against the σ-finite measure $\mathbf{N}$ as in (4.7), using Fubini and the monotonicity of the blanket times, yields

$$\mathbf{P}_{\bar{\varrho}}^{e,\mathcal{P}}\left(\tau_{\mathrm{bl}}^{e,\mathcal{P}}(\varepsilon-) = \tau_{\mathrm{bl}}^{e,\mathcal{P}}(\varepsilon+)\right) = 1,$$

ℓ-a.e. ε, $\mathbf{N}$-a.e. $(e,\mathcal{P})$, where $\mathbf{P}_{\bar{\varrho}}^{e,\mathcal{P}}$ denotes the law of $X^{e,\mathcal{P}}$ started from $\bar{\varrho}$. The rest of the argument relies on impoving such a statement to hold globally $\varepsilon \in (0,1)$.

In the transformed "glued" metric space $\mathcal{M}_{\Theta_b(e,\mathcal{P})}$, the Brownian motion admits $\mathbf{P}_{\bar{\varrho}}^{\Theta_b(e,\mathcal{P})}$-a.s. jointly continuous local times $(\sqrt{b}L_{b^{-3/2}t}(x))_{x\in\mathcal{M}_{e,\mathcal{P}},t\geq 0}$. This is enough to infer that, for every $\varepsilon \in (0,1)$ and $b > 1$, the continuity of the ε-blanket time variable of $\mathcal{M}_{e,\mathcal{P}}$ is equivalent (in law) to the continuity of the $b^{-1}\varepsilon$-blanket time variable of $\mathcal{M}_{\Theta_b(e,\mathcal{P})}$, and consequently as in the proof of Proposition 4.1.5, applying our scaling argument implies

$$\mathbf{P}_{\bar{\varrho}}^{e,\mathcal{P}}\left(\tau_{\mathrm{bl}}^{e,\mathcal{P}}(\varepsilon-) = \tau_{\mathrm{bl}}^{e,\mathcal{P}}(\varepsilon+)\right) = 1,$$

$\mathbf{N}$-a.e. $(e,\mathcal{P})$. Recall that, conditional on C_1, $\mathcal{M} \overset{(d)}{=} \mathcal{M}^{(C_1)}$, where C_1 is the length of the longest excursion of the process defined in (4.13), which is distributed as a tilted excursion of that length. Then, applying again our scaling argument as in the end of the proof of Proposition 4.1.5, conditional on C_1, we deduce

$$\mathbf{P}_{\varrho}^{\mathcal{M}}\left(\tau_{\mathrm{bl}}^{\mathcal{M}}(\varepsilon-) = \tau_{\mathrm{bl}}^{\mathcal{M}}(\varepsilon+)\right) = 1,$$

$\mathbf{N}_{C_1}$-a.e. $(e,\mathcal{P})$, where $\mathbf{N}_l$ is the version of $\mathbf{N}$ defined in (4.26) conditional on $\{L = l\}$. It was shown in (4.29) that the canonical measure $\mathbf{N}_{C_1}^{0,\lambda}$ is absolutely continuous with respect to $\mathbf{N}_{C_1}$, therefore the above also yields that conditional on C_1, $\mathbf{N}_{C_1}^{0,\lambda}$-a.e. $(e,\mathcal{P})$, $\tau_{\mathrm{bl}}^{\mathcal{M}}(\varepsilon)$ is continuous at ε, $\mathbf{P}_{\varrho}^{\mathcal{M}}$-a.s.

Proof of Theorem 4.0.4. Fix $\varepsilon \in (0,1)$. Here, for a particular real value of $\lambda \in \mathbb{R}$ and conditional on C_1,

$$\mathbb{P}_{\varrho}(\cdot) := \int \mathbf{P}_{\varrho}^{\mathcal{M}}(\cdot)\mathbf{N}_{C_1}^{0,\lambda}(d(e,\mathcal{P})),$$

formally defines the annealed measure for suitable events. Given the continuity of $\tau_{\mathrm{bl}}^{\mathcal{M}}(\varepsilon)$ at ε, $\mathbf{P}_{\varrho}^{\mathcal{M}}$-a.s. and Proposition 4.2.5, the desired annealed convergence follows by applying Theorem 3.1.2 exactly in the same manner as we did in the proof of Theorem 4.0.3 in the end of Subsection 4.1.1.

$\square$

4.3 Critical random graph with prescribed degrees

Let $\mathrm{G}_{n,d}$ denote the space of all simple graphs labeled by $[n]$ such that the i-th vertex has degree d_i, $i \geq 1$, for $1 \leq i \leq n$. We denote the vector $(d_i : i \in [n])$ of the prescribed degree sequence by d, where $\ell_n := \sum_{i\in[n]} d_i$ is assumed even. Write $\bar{\mathrm{G}}_{n,d}$ for $\mathrm{G}_{n,d}$ with the difference that we allow self-loops as well as multiple edges between the same pair of vertices. Then, the configuration model is the random multigraph in $\bar{\mathrm{G}}_{n,d}$ constructed as follows. Assign each vertex i with d_i half-edges, labelling them arbitrarily by $1, ..., \ell_n$. The multigraph $\mathrm{M}^n(d)$ produced by a uniform random pairing of the half-edges to create full edges is called the configuration model. In particular, we look at prescribed degree sequences that satisfy the following assumption.

Assumption 3. *Let D_n be a random variable with distribution given by*

$$P(D_n = i) = \frac{\#\{j : d_j = i\}}{n}.$$

In other words, D_n has the law of the degree of a vertex chosen uniformly at random from $[n]$. Suppose that $D_n \xrightarrow{(d)} D$, for a limiting random variable D such that $P(D = 1) > 0$. Moreover, assume the following as $n \to \infty$,

i) Convergence of third moments: $E(D_n^3) \to E(D^3) < \infty$,

ii) Scaling critical window: $\frac{E(D_n(D_n-1))}{E(D_n)} = 1+\lambda n^{-1/3}+o(n^{-1/3})$, for some $\lambda \in \mathbb{R}$. In particular, $E(D^2) = 2E(D)$.

Remark. *We remark here that the configuration model with random i.i.d. degrees sampled from a distribution with $E(D^3) < \infty$ treated in [65], meets the assumptions introduced above. More generally, ii) is assumed for $\lambda = 0$ and corresponds* *to the critical case, i.e. if $E(D^2) < 2E(D)$ there is no giant component with probability tending to 1, as $n \to \infty$. In addition, $E(D^2) > 2E(D)$ sees the emergence of a unique giant component with probability tending to 1, as $n \to \infty$, see [89].*

Write $c = (c_1, c_2, c_3) \in \mathbb{R}^3_+$ and define a Brownian motion with parabolic drift by

$$B_t^{c,\lambda} := \frac{\sqrt{c_2}}{c_1} B_t + \lambda t - \frac{c_2 t^2}{2c_1^3}, \qquad t \geq 0, \tag{4.30}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. The most general result under minimum assumptions for the joint convergence of the component sizes and the corresponding surpluses was proven in [42]. Fix $\lambda \in \mathbb{R}$ and let $(M_i^n)_{i\geq 1}$ and $(R_i^n)_{i\geq 1}$ denote the sequence of the sizes and surpluses of the components of $\mathrm{M}^n(d)$ respectively.

Theorem 4.3.1 (**Dhara, van der Hofstad, van Leeuwaarden, Sen** [42])**.** *As $n \to \infty$,*

$$\left(n^{-2/3}(M_i^n)_{i\geq 1}, (R_i^n)_{i\geq 1}\right) \longrightarrow ((M_i^{c_D})_{i\geq 1}, (R_i^{c_D})_{i\geq 1})$$

in distribution, where the convergence of the first sequence takes place in $\ell^2_\downarrow$. For the second sequence, the convergence takes place in the product topology.

The limiting sequence $(M_i^{c_D})_{i\geq 1}$ is distributed as the ordered sequence of lengths of excursions of the process $(B_t^{c_D,\lambda})_{t\geq 0}$ reflected at its minimum, where c_D has coordinates depending only on the first three moments of D and are given exactly by $c_1^D = E(D)$, $c_2^D = E(D^3)E(D) - (E(D^2))^2$ and $c_3^D = 1/E(D)$. Drawing the graph of the reflected process, scattering points on the plane according to a Poisson with rate c_3^D and keeping only those that fell between the x-axis and the function, describes $R_i^{c_D}$ as the number of those points that fell under the excursion with length $M_i^{c_D}$.

In Section 4.2, we introduced $\mathcal{M}^{(\zeta)}$ as the random real tree coded by a tilted Brownian excursion of length ζ to which a number of point identifications to create cycles is added. The number of point identifications is a Poisson random variable with mean given by the area under the tilted excursion. Given that number, say $k \geq 0$, x_i is picked with a density proportional to the height of the tilted excursion in an i.i.d. fashion for every $1 \leq i \leq k$ and y_i is picked uniformly from the path that connects the root to x_i. Then, x_i and y_i are identified. Let $\mathcal{M}^{(\zeta,c_3)}$ denote

$\mathcal{M}^{(\zeta)}$ if the number of point identifications is instead Poisson with mean given by the area under the tilted excursion multiplied by c_3, i.e. $c_3 \int_0^\zeta \tilde{e}(u)du$.

Then, the limit of the largest connected component of the configuration model in the scaling critical window, say $\mathrm{M}_1^n(d)$, can be written as a scalar multiple of $\mathcal{M}^{(M_1^{c_D}, c_3^D)}$, where $M_1^{c_D}$ is distributed according to the length of the longest excursion of the Brownian motion with parabolic drift with coefficients dependent on c_D as defined in (4.30). This statement is made precise as a simplified version of [23, Theorem 2.4], which we quote. As $n \to \infty$,

$$\left(n^{-2/3} M_1^n, \left(V(\mathrm{M}_1^n(d)), n^{-1/3} d_{\mathrm{M}_1^n(d)}, \varrho^n\right)\right) \longrightarrow \left(M_1^{c_D}, \left(\mathcal{M}_D, \frac{c_1^D}{\sqrt{c_2^D}} d_{\mathcal{M}_D}, \varrho\right)\right), \tag{4.31}$$

in distribution, where conditional on $M_1^{c_D}$, $\mathcal{M}_D \stackrel{(d)}{=} \mathcal{M}^{(M_1^{c_D}, c_3^D)}$. Actually [23, Theorem 2.4] holds also by considering the largest connected component of a uniform element of $\mathrm{G}_{n,d}$ with d satisfying the minimum conditions in Assumption 3. The limit of the maximal components of $G(n, n^{-1} + \lambda n^{-4/3})$ can be recovered by considering D_{ER} to be a mean 1 Poisson random variable ($c_1^{D_{\mathrm{ER}}} = c_2^{D_{\mathrm{ER}}} = c_3^{D_{\mathrm{ER}}} = 1$) by noticing the following two facts. By Poisson approximation to binomial, the random degree sequence of $G(n, n^{-1} + \lambda n^{-4/3})$ satisfies Assumption 3 with limiting random variable D_{ER}. Moreover, conditional on the event where the random degree sequence is equal to d, $G(n, n^{-1} + \lambda n^{-4/3})$ is uniformly distributed over $\mathrm{G}_{n,d}$.

We turn now our interest to how to sample uniformly a connected component with a given degree sequence $(\tilde{d}_i : i \in [\tilde{m}])$ that satisfies the following assumption.

Assumption 4. *Let $\tilde{d}_1 = 1$, and $\tilde{d}_i \geq 1$, for every $1 \leq i \leq \tilde{m}$. There exists a probability mass function $(\tilde{p}_i)_{i \geq 1}$ with the properties*

$$\tilde{p}_1 > 0, \qquad \sum_{i \geq 1} i\tilde{p}_i = 2, \qquad \sum_{i \geq 1} i^2 \tilde{p}_i < \infty,$$

such that

$$\frac{1}{\tilde{m}} \#\{j : \tilde{d}_j = i\} \to \tilde{p}_i, \ \textit{for all } i \geq 1, \ \textit{and} \qquad \frac{1}{\tilde{m}} \sum_{i \geq 1} \tilde{d}_i^2 \to \sum_{i \geq 1} i^2 \tilde{p}_i.$$

In particular, $\max_{1 \leq 1 \leq \tilde{m}} \tilde{d}_i = o(\sqrt{m})$.

For a given rooted plane tree θ with root ϱ, let $\mathrm{c}(\theta) = (\mathrm{c}_v(\theta))_{v\in\theta}$, where $\mathrm{c}_v(\theta)$ gives the number of children of v in θ and let $\mathrm{s}(\theta) = (\mathrm{s}_i(\theta))_{i\geq 0}$ be the empirical children distribution (ECD) of θ, i.e. $\mathrm{s}_i(\theta) := \#\{v \in \theta : c_v(\theta) = i\}$, for every $i \geq 0$. Note that $\mathrm{s}_0(\theta) = \#\mathcal{L}(\theta)$ gives exactly the number of leaves of θ. Now, given a sequence of integers $\mathrm{s} = (\mathrm{s}_i)_{i\geq 0}$, note that there exists a finite plane tree θ with $\mathrm{s}(\theta) = \mathrm{s}$ if and only if $\mathrm{s}_0 \geq 1$, $\mathrm{s}_i \geq 0$ for every $i \geq 1$, and

$$\sum_{i\geq 0} \mathrm{s}_i = 1 + \sum_{i\geq 1} i\mathrm{s}_i < \infty.$$

Given s, let $\mathbf{T}_\mathrm{s}$ denote the collection of all plane trees having ECD s. Let $x, y \in \mathcal{L}(\theta)$. We say that the ordered pair of leaves (x, y) is admissible if $\mathrm{par}(x) <_{\mathrm{DF}} \mathrm{par}(y)$, i.e. if the parent of x is explored before the parent of y during a depth-first search of θ, and if $\mathrm{gpar}(y) \in [[\varrho, \mathrm{gpar}(x)]]$, i.e. if the grandparent of y belongs to the ancestral line that connects the root to the grandparent of x. Let $(\mathbf{A}(\theta), <<)$ denote the collection of pairs of admissible leaves of θ endowed with the linear order $<<$ that declares $(x_1, y_1) << (x_2, y_2)$ if and only if $x_1 <_{\mathrm{DF}} x_2$, or if $x_1 = x_2$ and $y_1 <_{\mathrm{DF}} y_2$. For $k \geq 1$, we denote by $\mathbf{A}_k(\theta)$ the collection of admissible k-tuples of $2k$ distinct leaves and by $\mathbf{T}^k_\mathrm{s}$ the pairs (θ, z) for which $\theta \in \mathbf{T}_\mathrm{s}$ and $\mathrm{z} \in \mathbf{A}_k(\theta)$. Finally, for a rooted plane tree θ and $\mathrm{z} = \{(x_1, y_1), ..., (x_k, y_k)\} \in \mathbf{A}_k(\theta)$, we denote by $L(\theta, \mathrm{z})$ the rooted plane tree obtained from θ performing the following operation. Delete x_i, y_i together with the edges adjacent to them for each $i = 1, ..., k$ and add an edge between $\mathrm{par}(x_i)$ and $\mathrm{par}(y_i)$. We equip $L(\theta, \mathrm{z})$ with the shortest path distance and the uniform probability measure on its set of vertices.

We will work with connected graphs with a fixed surplus, so suppose that

$$\sum_{i\in[\tilde{m}]} \tilde{d}_i = 2(\tilde{m} - 1) + 2k,$$

for some fixed $k \geq 0$. Under Assumption 4, the lowest labeled vertex has one descendant and for the remaining vertices $2, ..., \tilde{m}$ we form the children sequence $\mathrm{c} := (\mathrm{c}_i)_{i=2}^{\tilde{m}+2k}$ via $\mathrm{c}_i := \tilde{d}_i - 1$, for every $2 \leq i \leq \tilde{m}$, and $\mathrm{c}_{\tilde{m}+j} := 0$, for every $1 \leq j \leq 2k$. Observe that

$$\sum_{i=2}^{\tilde{m}+2k} c_i = \sum_{i=2}^{\tilde{m}} \tilde{d}_i - (\tilde{m} - 1) = (\tilde{m} - 1 + 2k) - 1, \tag{4.32}$$

and thus c can be seen as representing a children sequence for a plane tree with $m := \tilde{m} - 1 + 2k$ vertices. Write $\mathrm{s} := (\mathrm{s}_i)_{i\geq 0}$ for its ECD. The following lemma is due to Bhamidi and Sen.

Lemma 4.3.2 (**Bhamidi, Sen** [23]). *To generate uniformly a connected graph with prescribed degree sequence $\tilde{d}$ satisfying Assumption 4,*

i) Generate first $(\tilde{T}_{\mathrm{s}}, \tilde{Z})$ uniformly from $\mathbf{T}_{\mathrm{s}}^k$. If $\tilde{Z} = \{(x_1, y_1), ..., (x_k, y_k)\}$ with

$$(x_1, y_1) << ... << (x_k, y_k),$$

label x_i as $\tilde{m} + 2i - 1$ and y_i as $\tilde{m} + 2i$, $1 \leq i \leq k$. Label the rest of the $\tilde{m} - 1$ vertices uniformly using the remaining labels $2, ..., \tilde{m}$ so that in the resulting labeled tree the vertex j has exactly $\tilde{d}_j - 1$ children. Call this labeled tree $\tilde{T}_{\mathrm{s}}^{\mathrm{lb}}$.

ii) Construct $L(\tilde{T}_{\mathrm{s}}^{\mathrm{lb}}, \tilde{Z})$, attach a vertex labeled 1 to the root and forget about the planar order and the root. Call $\mathcal{G}$ the resulting graph.

Then, $\mathcal{G}$ is distributed uniformly over the set of connected graphs with prescribed degree sequence $\tilde{d}$.

Proof. Fix a connected graph with prescribed degree sequence $\tilde{d}$. Recall that $\tilde{d}_1 = 1$. Remove vertex 1 with the only adjacent vertex to it and replace it with a root, which we call x_1 and denote the resulting graph by G. We construct a labeled plane tree from G using a depth-first procedure that in each step deletes the edges that create surplus while adding two leaves to the disconnected vertices. To be more precise, starting from the root x_1 on the top of the stack, set its status as explored and the status of its neighbors as seen. Then, shuffle all its neighbors uniformly, pick the leftmost one, call it x_2 and place it on the top of the stack declaring it open. We proceed inductively as follows. At step k, for some $k \geq 2$, if x_k is on the top of the stack, we set its status as explored and the status of its neighbors as seen only if none of them have been previously seen while shuffling them uniformly and declaring the leftmost one, x_{k+1}, open.

Suppose that, before exploring x_k, we found r_0 edges that create surplus and that at x_k we found r_1 many new edges that create surplus, say $e_1, ..., e_{r_1}$, where $e_i = x_k u_i$, $i = 1, ..., r_1$, and $u_1 <_{\mathrm{DF}} ... <_{\mathrm{DF}} u_{r_1}$. For $i = 1, ..., r_1$, delete e_i and create two leaves labeled $\tilde{m} + 2r_0 + 2i - 1$ and $\tilde{m} + 2r_0 + 2i$ in such a way that $x_k = \mathrm{par}(\tilde{m} + 2r_0 + 2i - 1)$ and $u_i = \mathrm{par}(\tilde{m} + 2r_0 + 2i)$. Shuffle the neighbors of

x_k, including the newly created leaves, set their status as seen and the status of x_k as explored, move to the leftmost one and call it x_{k+1}. Note that we do not set the status of $\tilde{m} + 2r_0 + 2i$, $i = 1, ..., r_1$ as seen since it is not a neighbor of x_k.

Let $T(G)^{\mathrm{lb}}$ denote the labeled depth-first tree recovered and set

$$\mathrm{z} = \{(\tilde{m}+1, \tilde{m}+2), ..., (\tilde{m}+2k-1, \tilde{m}+2k)\}.$$

Observe that $T(G)^{\mathrm{lb}}$ has always children sequence given by (4.32) and z contains k admissible pairs of leaves. Therefore, $(T(G)^{\mathrm{lb}}, \mathrm{z}) \in \mathbf{T}_{\mathrm{s}}^{k}$. Now,

$$P\left(L(\tilde{T}_{\mathrm{s}}^{\mathrm{lb}}, \tilde{Z}) = G\right) = \sum_{(T(G)^{\mathrm{lb}},\mathrm{z})} P\left((\tilde{T}_{\mathrm{s}}^{\mathrm{lb}}, \tilde{Z}) = (T(G)^{\mathrm{lb}}, \mathrm{z})\right),$$

where the sum is taken over the set of all labeled elements of $\mathbf{T}_{\mathrm{s}}^{k}$ that can be obtained through the depth-first algorithm outlined above. The aforementioned set has cardinality $\prod_{i=2}^{\tilde{m}}(\tilde{d}_i - 1)!$ due to the uniform shuffling we commit in each step. Recalling how $(\tilde{T}_{\mathrm{s}}^{\mathrm{lb}}, \tilde{Z})$ is generated, we complete the calculation we started. Namely,

$$\begin{aligned} P\left(L(\tilde{T}_{\mathrm{s}}^{\mathrm{lb}}, \tilde{Z}) = G\right) &= \sum_{(T(G)^{\mathrm{lb}},\mathrm{z})} P\left((\tilde{T}_{\mathrm{s}}^{\mathrm{lb}}, \tilde{Z}) = (T(G)^{\mathrm{lb}}, \mathrm{z})\right) \\ &= \prod_{i=2}^{\tilde{m}}(\tilde{d}_i - 1)! \cdot \frac{1}{|\mathbf{T}_{\mathrm{s}}^{k}|} \cdot \frac{1}{(\mathrm{s}_0 - 2k)! \prod_{i\geq 1} \mathrm{s}_i!}. \end{aligned}$$

Now, since this probability is a function of $\tilde{m}$ only, the desired result follows.

□

We use ϱ^m to denote the root of $\tilde{T}_{\mathrm{s}}$. We denote by $\tilde{C}_m^{\mathrm{s}} = (\tilde{C}_m^{\mathrm{s}}(i) : 0 \leq i \leq 2m)$ the contour function of $\tilde{T}_{\mathrm{s}}$, the random tree generated according to *i)* in the statement of Lemma 4.3.2, and by

$$\tilde{C}_{(m)}^{\mathrm{s}}(s) = \frac{\tilde{C}_m^{\mathrm{s}}(2ms)}{\sqrt{m}}, \qquad 0 \leq s \leq 1,$$

its normalized contour function as well. In the next lemma we show that, for some $\alpha > 0$, the sequence $||\tilde{C}_{(m)}^{\mathrm{s}}||_{H_\alpha}$ of Hölder norms is tight.

Lemma 4.3.3. *There exists $\alpha > 0$, such that*

$$\lim_{M\to\infty} \liminf_{m\to\infty} P\left(\sup_{0\le s\neq t\le 1} \frac{|\tilde{C}^{\mathrm{s}}_{(m)}(s) - \tilde{C}^{\mathrm{s}}_{(m)}(t)|}{|t-s|^{\alpha}} \le M\right) = 1. \tag{4.33}$$

Proof. From the definition of $(\tilde{T}_{\mathrm{s}}, \tilde{Z})$, it is clear that

$$P(\tilde{T}_{\mathrm{s}} = \theta) = \frac{|\mathbf{A}_k(\theta)|}{|\mathbf{T}^k_{\mathrm{s}}|},$$

for any $\theta \in \mathbf{T}_{\mathrm{s}}$, i.e. $\tilde{T}_{\mathrm{s}}$ is a random tree that has "tilted" distribution which is biased in favor of trees with large collection of admissible k-tuples between $2k$ distinct leaves. Hence, for any $f : C([0,1], \mathbb{R}_+) \to \mathbb{R}_+$ bounded and continuous function,

$$E[f(\tilde{C}^{\mathrm{s}}_{(m)})] = \frac{E[f(C^{\mathrm{s}}_{(m)})|\mathbf{A}_k(T_{\mathrm{s}})|]}{E[|\mathbf{A}_k(T_{\mathrm{s}})|]}, \tag{4.34}$$

where T_{s} is a uniform plane tree having ECD s, which is specified by the children sequence described in (4.32). Here, $C^{\mathrm{s}}_{(m)}$ is the normalized contour function that encodes T_{s}. Note that when $\tilde{d}$ satisfies Assumption 4, s satisfies the following:

$$\sum_{i\ge 0} \mathrm{s}_i = m, \qquad \frac{\mathrm{s}_i}{m} \to p_i, \text{ for all } i \ge 0, \text{ and } \qquad \frac{1}{m}\sum_{i\ge 0} i^2 \mathrm{s}_i \to \sum_{i\ge 0} i^2 p_i.$$

In particular, $\max\{i : \mathrm{s}_i \neq 0\} = o(\sqrt{m})$. Also, $p := (p_i)_{i\ge 0}$ is a probability mass function with $p_i := \tilde{p}_{i+1}$, for every $i \ge 0$, and therefore it has the properties

$$p_0 > 0, \qquad \sum_{i\ge 0} i p_i = 1, \qquad \sum_{i\ge 0} i^2 p_i < \infty. \tag{4.35}$$

By (4.34) along with the Cauchy-Schwarz inequality, we deduce

$$P\left(\sup_{0\le s\neq t\le 1}\frac{|\tilde{C}^{\mathrm{s}}_{(m)}(s)-\tilde{C}^{\mathrm{s}}_{(m)}(t)|}{|t-s|^{\alpha}}\ge M\right)=\frac{E\left[\mathbb{1}_{\left\{\sup_{0\le s\neq t\le 1}\frac{|C^{\mathrm{s}}_{(m)}(s)-C^{\mathrm{s}}_{(m)}(t)|}{|t-s|^{\alpha}}\ge M\right\}}|\mathbf{A}_k(T_{\mathrm{s}})|\right]}{E[|\mathbf{A}_k(T_{\mathrm{s}})|]}$$

$$\le\frac{P\left(\sup_{0\le s\neq t\le 1}\frac{|C^{\mathrm{s}}_{(m)}(s)-C^{\mathrm{s}}_{(m)}(t)|}{|t-s|^{\alpha}}\ge M\right)^{1/2}\left(E\left[\left(\frac{|\mathbf{A}_k(T_{\mathrm{s}})|}{\mathrm{s}_0^k m^{k/2}}\right)^2\right]\right)^{1/2}}{E\left[\frac{|\mathbf{A}_k(T_{\mathrm{s}})|}{\mathrm{s}_0^k m^{k/2}}\right]}. \tag{4.36}$$

Using [23, Lemma 6.3(ii)], we have that

$$\sup_{m\ge 1}E\left[\left(\frac{|\mathbf{A}_k(T_{\mathrm{s}})|}{\mathrm{s}_0^k m^{k/2}}\right)^2\right]\le\frac{1}{k!}\sup_{m\ge 1}E\left[\left(\frac{|\mathbf{A}(T_{\mathrm{s}})|}{\mathrm{s}_0\sqrt{m}}\right)^{2k}\right]<\infty,$$

for every $k\ge 1$. Furthermore, using [23, Lemma 6.3(iii)], [23, Lemma 6.3(v)] together with the uniform integrability from above, we conclude that

$$E\left[\frac{|\mathbf{A}_k(T_{\mathrm{s}})|}{\mathrm{s}_0^k m^{k/2}}\right]\to\left(\frac{p_0\sigma}{2}\right)^k E\left[\left(\int_0^1 2e(u)du\right)^k\right]>0,$$

as $m\to\infty$, where $(e(t))_{t\in[0,1]}$ is a normalized Brownian excursion and $\sigma^2=\sum_{i\ge 0}i^2p_i-1$. It only remains to deal with the quantity $P(||C^{\mathrm{s}}_{(m)}||_{H_\alpha}\ge M)$ that appears on the right-hand side of (4.36). It turns out that plane trees chosen uniformly from $\mathbf{T}_{\mathrm{s}}$ are related to Galton-Watson trees by a simple conditioning. The uniform distribution on $\mathbf{T}_{\mathrm{s}}$ coincides with the distribution of a Galton-Watson tree θ with offspring distribution $\mu:=(\mu_i)_{i\ge 0}$, which must satisfy $\mu_i>0$ if $\mathrm{s}_i>0$, conditioned on the event $\cap_{i\ge 0}\{\mathrm{s}_i(\theta)=\mathrm{s}_i\}$. Take $\mu=p$ as in (4.35) to be the critical offspring distribution with finite variance of a Galton-Watson tree θ. Then, if P_p is the probability distribution of θ,

$$P\left(||C^{\mathrm{s}}_{(m)}||_{H_\alpha}\ge M\right)=P_p\left(||C_{(m)}||_{H_\alpha}\ge M|\mathrm{s}_i(\theta)=\mathrm{s}_i,\forall i\ge 0\right),$$

where $||C_{(m)}||_{H_\alpha}$ denotes the α-Hölder norm of the normalized contour function $C_{(m)}$ that encodes θ. The proof is completed as a result of Theorem 4.1.3.

□

In Lemma 4.3.2, we saw that $\mathcal{G}$, a uniformly chosen connected graph with prescribed degree sequence $\tilde{d}$ satisfying Assumption 4, is distributed as $L(\tilde{T}_{\mathrm{s}}^{\mathrm{lb}}, \tilde{Z})$, where $(\tilde{T}_{\mathrm{s}}^{\mathrm{lb}}, \tilde{Z})$ is a uniform labeled element of $\mathbf{T}_{\mathrm{s}}^{k}$. Recall that to obtain $L(\tilde{T}_{\mathrm{s}}^{\mathrm{lb}}, \tilde{Z})$ from $(\tilde{T}_{\mathrm{s}}^{\mathrm{lb}}, \tilde{Z})$, where $\tilde{Z} = \{(x_i, y_i), ..., (x_k, y_k)\}$ with $(x_1, y_1) << ... << (x_k, y_k)$, for every pair of admissible leaves, we add an edge between $\mathrm{par}(x_i)$ and $\mathrm{par}(y_i)$, deleting x_i, y_i and the two edges incident to them, for $1 \leq i \leq k$. The resistance on $L(\tilde{T}_{\mathrm{s}}^{\mathrm{lb}}, \tilde{Z})$ between two vertices is smaller than the total length of the path between them on $\tilde{T}_{\mathrm{s}}^{\mathrm{lb}}$. This observation together with Lemma 4.3.3 is enough to establish equicontinuity of the rescaled local times $(L_t^{\mathcal{G}}(x))_{x \in V(\mathcal{G}), t \geq 0}$ of the simple random walk on $\mathcal{G}$ under the annealed law. Since the proof relies heavily on arguments that are present in the proof of Proposition 4.1.4 we omit it.

Lemma 4.3.4. *For every $\varepsilon > 0$ and $T > 0$,*

$$\lim_{\delta \to 0} \limsup_{m \to \infty} \mathbb{P}_{\varrho^m} \left(\sup_{\substack{y,z \in V(\mathcal{G}): \\ m^{-1/2} R_{\mathcal{G}}(y,z) < \delta}} \sup_{t \in [0,T]} m^{-1/2} |L^{\mathcal{G}}_{m^{3/2}t}(y) - L^{\mathcal{G}}_{m^{3/2}t}(z)| \geq \varepsilon \right) = 0.$$

Assume that d satisfies Assumption 3 with limiting random variable D, and let D^* denote its size-biased distribution given by

$$p_i^* := P(D^* = i) = \frac{iP(D = i)}{E(D)}, \qquad i \geq 1.$$

Then, for $\mathrm{M}_1^n(d)$, the largest connected component of the configuration model $\mathrm{M}^n(d)$, if we denote by $\xrightarrow{\mathrm{P}}$ convergence in probability,

$$\frac{\#\{v \in \mathrm{M}_1^n(d) : d_v = i\}}{|V(\mathrm{M}_1^n(d))|} \xrightarrow{\mathrm{P}} p_i^*, \qquad \frac{1}{|V(\mathrm{M}_1^n(d))|} \sum_{v \in \mathrm{M}_1^n(d)} d_v^2 \xrightarrow{\mathrm{P}} \sum_{i \geq 1} i^2 p_i^* < \infty, \quad (4.37)$$

$$P(\mathrm{M}_1^n(d) \text{ is simple}) \to 1, \quad (4.38)$$

for all $i \geq 1$. For a justification of (4.37) and (4.38), see [23, Proposition 8.2]. Note that $P(D = 1) > 0$ under Assumption 3, and hence $p_1^* > 0$. Furthermore, under Assumption 3,

$$\sum_{i \geq 1} i p_i^* = \frac{E(D^2)}{E(D)} = 2,$$

and this shows, along with (4.37), that $(d_v : v \in \mathrm{M}_1^n(d))$ satisfies Assumption 4 (after a possible relabelling of the vertices) with limiting probability mass function $p^* := (p_i^*)_{i\geq 1}$. Let $\mathcal{P}$ denote the partition of $\mathrm{M}^n(d)$ into different components. Conditional on the event $\{\mathrm{M}_1^n(d) \text{ is simple}\} \cap \{\mathcal{P} = P\}$, $\mathrm{M}_1^n(d)$ is uniformly distributed over the set of simple, connected graphs with degree sequence decided by the partition P, see [62, Proposition 7.7]. Since $P(\mathrm{M}_1^n(d) \text{ is a multigraph}) \to 0$ by (4.38), imitating the argument used in the proof of Proposition 4.2.5, the result below follows as a combination of Theorem 4.3.1 and Lemma 4.3.4.

Proposition 4.3.5. *Under Assumption 3, for every $\varepsilon > 0$ and $T > 0$, the rescaled local times of the simple random walk on $\mathrm{M}_1^n(d)$ are equicontinuous under the annealed law, i.e.*

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}_{\varrho^n}\left(\sup_{\substack{y,z\in V(\mathrm{M}_1^n(d)):\\ n^{-1/3}R_{\mathrm{M}_1^n(d)}(y,z)<\delta}}\ \sup_{t\in[0,T]} n^{-1/3}|L_{nt}^n(y) - L_{nt}^n(z)| \geq \varepsilon\right) = 0.$$

4.3.1 Convergence of the walks

Croydon [36] used regular resistance forms to describe the scaling limit of the associated random walks on scaling limits of sequences of spaces equipped with resistance metrics and measures provided that they converge with respect to a suitable Gromov-Hausdorff topology, and under the assumption that a non-explosion condition is satisfied. For families of random graphs that are nearly trees and their scaling limit can be described as a tree "glued" at a finite number of pairs of points, a useful corollary of [36, Theorem 1.2] combined with [36, Proposition 8.4] yields the convergence of the processes associated with the fused spaces.

To see that the conclusion of [36, Proposition 8.4] holds, recall that under Assumption 3, jointly with Theorem 4.3.1,

$$\left(V(\mathrm{M}_1^n(d)), n^{-1/3}d_{\mathrm{M}_1^n(d)}, \varrho^n\right) \longrightarrow \left(\mathcal{M}_D, \frac{c_1^D}{\sqrt{c_2^D}}d_{\mathcal{M}_D}, \varrho\right),$$

as $n\to\infty$ in the Gromov-Hausdorff sense. Let $\mathcal{P}$ denote the partition of $\mathrm{M}^n(d)$ into different components. Conditional on the event $\{\mathrm{M}_1^n(d) \text{ is simple}\} \cap \{\mathcal{P} = P\}$, $\mathrm{M}_1^n(d)$ is uniformly distributed over the set of simple, connected graphs with degree sequence decided by the partition P, and therefore the convergence above is valid with $\mathrm{M}_1^n(d)$ replaced by $L(\tilde{T}_{\mathrm{s}}, \tilde{Z})$, see Lemma 4.3.2 for its construction.

If $\tilde{Z} = \{(x_1, y_1), ..., (x_{R_1^n}, y_{R_1^n})\}$ with $(x_1, y_1) << ... << (x_{R_1^n}, y_{R_1^n})$, let $D(\tilde{T}_s, \tilde{Z})$ be the space obtained by fusing x_i and $\text{gpar}(y_i)$, $1 \le j \le R_1^n$, endowed with the graph distance and the push-forward of the uniform probability measure on $\tilde{T}_s$, and observe that

$$d_{\tilde{\mathbb{K}}}(L(\tilde{T}_s, \tilde{Z}), D(\tilde{T}_s, \tilde{Z})) \le 5R_1^n.$$

Thus, jointly with Theorem 4.3.1, the convergence above is valid with $\mathrm{M}_1^n(d)$ replaced with the "glued" tree $D(\tilde{T}_s, \tilde{Z})$, and since $\mathcal{M}_D$ is also a "glued" tree, this shows that the conclusion of [36, Proposition 8.4] is valid.

It remains to show that

$$\lim_{r\to\infty} \liminf_{n\to\infty} P\left(n^{-1/3} d_{\mathrm{M}_1^n(d)}(\varrho^n, B_n(\varrho^n, r)^c) \ge \lambda\right) = 1, \qquad \forall \lambda \ge 0.$$

To bound the last probability from below, intersect it with the event $n^{-1/3} D_1^n(d) \le r$, under which $B_n(\varrho^n, r)^c = \emptyset$. Indeed, the distance from $\varrho^n \in \mathrm{M}_1^n(d)$ to $\emptyset$ is $+\infty$, and therefore

$$\begin{aligned} &P\left(n^{-1/3} d_{\mathrm{M}_1^n(d)}(\varrho^n, B_n(\varrho^n, r)^c) \ge \lambda\right) \\ &\ge P\left(n^{-1/3} d_{\mathrm{M}_1^n(d)}(\varrho^n, B_n(\varrho^n, r)^c) \ge \lambda, n^{-1/3} D_1^n(d) \le r\right) \\ &= P(n^{-1/3} D_1^n(d) \le r), \end{aligned} \tag{4.39}$$

where $D_1^n(d) := \mathrm{diam}_{\mathrm{M}_1^n(d)}(\mathrm{M}_1^n(d))$. Letting $D_1(d) := \mathrm{diam}_{\mathcal{M}_D}(\mathcal{M}_D)$, since for the two (and any) metric spaces $\mathrm{M}_1^n(d)$ and $\mathcal{M}_D$,

$$|n^{-1/3} D_1^n(d) - D_1(d)| \le 2 d_{\tilde{\mathbb{K}}}(n^{-1/3} \mathrm{M}_1^n(d), \mathcal{M}_D),$$

the convergence $n^{-1/3} D_1^n(d) \to D_1(d)$ in distribution is immediate from (4.31). By this and (4.39),

$$\begin{aligned} \liminf_{n\to\infty} P\left(n^{-1/3} d_{\mathrm{M}_1^n(d)}(\varrho^n, B_n(\varrho^n, r)^c) \ge \lambda\right) &\ge \liminf_{n\to\infty} P(n^{-1/3} D_1^n(d) \le r) \\ &= P(D_1(d) \le r). \end{aligned}$$

As $r \to \infty$, the right-hand-side tends to 1, and this shows that the claimed non-explosion is fulfilled. As a consequence, we have the convergence of the processes associated with the fused spaces. It is possible to isometrically embed $(\mathrm{M}_1^n(d), d_{\mathrm{M}_1^n(d)})$, $n \ge 1$ and $(\mathcal{M}_D, d_{\mathcal{M}_D})$ into a common metric space (F, d_F), such

that

$$\mathbf{P}_{\varrho^n}^{\mathrm{M}_1^n(d)}\left((n^{-1/3}X^n_{\lfloor nt\rfloor})_{t\geq 0}\in\cdot\right)\to\mathbf{P}_{\varrho}\left((X_t^{\mathcal{M}_D})_{t\geq 0}\in\cdot\right), \tag{4.40}$$

weakly as probability measures in $D(\mathbb{R}_+, F)$.

4.3.2 Continuity of blanket times of Brownian motion on $\mathcal{M}_D$

In Section 4.2.1, we presented $\mathbb{N}^{t,\lambda}$, the inhomogeneous excursion (for excursions starting at time t) measure associated with a Brownian motion with parabolic drift as defined in (4.13). Denote by $\mathbb{N}_t^{c,\lambda}$ the excursion measure associated with $B^{c,\lambda}$ as defined in (4.30). Write $(e(u): 0\leq u\leq t)$ for the canonical process under $\mathbb{N}$. By the Cameron-Martin-Girsanov formula [92, Chapter IX, (1.10) Theorem], applied under $\mathbb{N}$,

$$\frac{d\mathbb{N}_0^{c,\lambda}}{d\mathbb{N}}=\exp\left(\frac{\sqrt{c_2}}{c_1}\int_0^t\gamma(u)de(u)-\frac{1}{2}\int_0^t\gamma^2(u)du\right),$$

where $\gamma(u):=\lambda-\frac{c_2}{c_1^3}u$ is the drift. On the set of excursions of length t, using integration by parts, we have that

$$\frac{\sqrt{c_2}}{c_1}\int_0^t\left(\lambda-\frac{c_2}{c_1^3}u\right)de(u)=\frac{c_2^{3/2}}{c_1^4}\int_0^t e(u)du,$$

a multiple of the area under the excursion of length t. So, the density becomes

$$\frac{d\mathbb{N}_0^{c,\lambda}}{d\mathbb{N}}=\exp\left(\frac{c_2^{3/2}}{c_1^4}\int_0^t e(u)du-\frac{1}{6}\left(\left(\frac{c_2}{c_1^3}t-\lambda\right)^3+\lambda^3\right)\right).$$

There is a corresponding probability measure $\mathbb{N}_{0,l}^{c,\lambda}:=\mathbb{N}_0^{c,\lambda}(\cdot|\tilde{L}=l)$, which for a Borel set B on the space of positive excursions of finite length, is determined by

$$\mathbb{N}_{0,l}^{c,\lambda}(\mathbb{1}_B)=\frac{\mathbb{N}_l\left(\exp\left(\frac{c_2^{3/2}}{c_1^4}\int_0^l e(u)du\right)\mathbb{1}_B\right)}{\mathbb{N}_l\left(\exp\left(\frac{c_2^{3/2}}{c_1^4}\int_0^l e(u)du\right)\right)}.$$

To determine $\mathbb{N}_0^{c,\lambda}(\tilde{L} \in dl)$, recall that $\mathbb{N}(L \in dl) = f_L(\lambda) = dl/\sqrt{2\pi l^3}$, $l \geq 0$, and therefore

$$\mathbb{N}_0^{c,\lambda}(\tilde{L} \in dl) = f_L(l) \exp\left(-\frac{1}{6}\left(\left(\frac{c_2}{c_1^3}l - \lambda\right)^3 + \lambda^3\right)\right) \mathbb{N}_l\left(\exp\left(\frac{c_2^{3/2}}{c_1^4}\int_0^l e(u)du\right)\right).$$

Let $\mathbf{N}_t^{c,\lambda}$ denote the canonical measure that first at time t picks a tilted Brownian excursion of a randomly chosen length l, and then independently of e chooses a number of points according to a Poisson random variable with mean $c_3\int_0^l e(t)dt$, which subsequently are distributed uniformly on the area under the graph of e. In comparison with (4.28), we characterize $\mathbf{N}_t^{c,\lambda}(d(e,\mathcal{P}))$ by setting:

$$\begin{aligned}
&\mathbf{N}_t^{c,\lambda}(de, |\mathcal{P}| = k, (dx_1, ..., dx_k) \in A_1 \times ... \times A_k) \\
&:= \int_0^\infty \mathbb{N}_0^{c,\lambda-t}(\tilde{L} \in dl)\mathbb{N}_{t,l}^{c,\lambda}(de) \exp\left(-c_3\int_0^l e(u)du\right) \frac{\left(c_3\int_0^l e(u)du\right)^k}{k!} \prod_{i=1}^k \frac{\ell(A_i \cap A_e)}{\ell(A_e)}.
\end{aligned} \tag{4.41}$$

It is easy to see that $\mathbf{N}_t^{c,\lambda}$ is absolutely continuous with respect to $\mathbf{N}$ as defined in (4.26). More specifically,

$$\frac{d\mathbf{N}_t^{c,\lambda}}{d\mathbf{N}} = \exp\left(1 - \frac{1}{6}\left(\lambda^3 + \left(\frac{c_2}{c_1^3}l - \lambda + t\right)^3\right)\right)\left(c_3\int_0^l e(u)du\right)^k. \tag{4.42}$$

Proof of Theorem 4.0.5. Applying our scaling argument as in Subsection 4.2.1 yields that conditional on $M_1^{c_D}$, $\mathbf{N}_{M_1^{c_D}}$-a.e. $(e,\mathcal{P})$, $\tau_{\text{bl}}^{\mathcal{M}_D}(\varepsilon)$ is continuous at ε, $\mathbf{P}_\varrho^{\mathcal{M}_D}$-a.s. In (4.42), it was shown that the canonical measure $\mathbf{N}_0^{c,\lambda}$ is absolutely continuous with respect to $\mathbf{N}$, therefore the above also yields that conditional on $M_1^{c_D}$, $\mathbf{N}_{0,M_1^{c_D}}^{c,\lambda}$-a.e. $(e,\mathcal{P})$, $\tau_{\text{bl}}^{\mathcal{M}_D}(\varepsilon)$ is continuous at ε, $\mathbf{P}_\varrho^{\mathcal{M}_D}$-a.s., where $\mathbf{N}_{t,l}^{c,\lambda}$ is the version of $\mathbf{N}_t^{c,\lambda}$ conditional on $\{\tilde{L} = l\}$. Fix $\varepsilon \in (0,1)$. Here, for a particular real value of $\lambda \in \mathbb{R}$ and conditional on $M_1^{c_D}$,

$$\mathbb{P}_\varrho(\cdot) := \int \mathbf{P}_\varrho^{\mathcal{M}_D}(\cdot)\mathbf{N}_{0,M_1^{c_D}}^{c,\lambda}(d(e,\mathcal{P})),$$

formally defines the annealed measure for suitable events. Given the continuity of $\tau_{\text{bl}}^{\mathcal{M}_D}(\varepsilon)$ at ε, $\mathbf{P}_\varrho^{\mathcal{M}_D}$-a.s. and Proposition 4.3.5, the desired annealed convergence follows by applying Theorem 3.1.2 exactly in the same manner as we did in the proof of Theorem 4.0.3 in the end of Subsection 4.1.1.

□

Chapter 5

Random walk in random environment on plane trees

In Section 5.1, we introduce the random walk in random environment on locally finite ordered trees as a resistor network with conductances and stationary reversible measure given in terms of its potential, while the rest of the section ties together the preliminary work done to yield the convergence of the random walks in random environments under Assumption 5, as a corollary of the main contribution of [36]. Finally, along with extending Seignourel's result in [95] to hold for a wider class of environments, we prove Theorem 5.4.2 and Theorem 5.5.4.

5.1 Set-up and main assumption

Let T be a locally finite ordered tree with a distinguished vertex ϱ. For each $u \in T$, we denote its children by $u_1, ..., u_{\xi(u)}$ and its parent by u_0. Note that $\xi(u) < \infty$, for every $u \in T$, since T was assumed to be locally finite. For each $u \in T$, let

$$N_u := \left\{ (\omega_{uu_i})_{i=0}^{\xi(u)} : \omega_{uu_i} > 0 \;\; \forall\, 0 \leq i \leq \xi(u) \text{ and } \sum_{i=0}^{\xi(u)} \omega_{uu_i} = 1 \right\},$$

where $\omega_{uu_i} : T \to (0,1)$ is a measurable function indexed by the directed edge connecting u to its neighbor u_i. Formally, N_u is the set of transition laws at u. We equip N_u with the weak topology on probability measures, which turns it into a Polish space. Let $\Omega := \prod_{u \in T} N_u$ equipped with the product topology that carries the Polish structure of N_u, and the corresponding Borel σ-algebra $\mathcal{F}$, which is the same as the σ-algebra generated by cylinder functions. For a probability measure P on $(\Omega, \mathcal{F})$, a random environment ω is an element of Ω that has law as P.

For each $\omega \in \Omega$, the random walk in the random environment (RWRE) ω is the time-homogeneous Markov chain $X = ((X_n)_{n\geq 0}, \mathbf{P}_\omega^u, u \in T)$ taking values on T with transition probabilities, for each $u \in T$, given by

$$(\mathbf{P}_\omega(X_{n+1} = u_i | X_n = u))_{i=0}^{\xi(u)} = (\omega_{uu_i})_{i=0}^{\xi(u)}. \tag{5.1}$$

Using the same terminology from the literature of RWRE, for $u \in T$, we refer to $\mathbf{P}_\omega^u$ as the quenched law of X started from u. For each non-root vertex $u \in T$, let $\vec{u}$ denote the parent of u. Then, the fraction $\varrho_{\vec{u}u} := \omega_{\vec{u}\vec{\vec{u}}}/\omega_{\vec{u}u}$ is well-defined for every node of T except the root and any of its children. Suppose that the marginals of ω are defined as the transition probabilities of a weighted random walk on T with conductances assigned on its (undirected) edge set $E(T)$. More specifically, for each $u \in T$, let

$$(\omega_{uu_i})_{i=0}^{\xi(u)} = \left(\frac{c(\{u, u_i\})}{c(\{u\})} : 0 \leq i \leq \xi(u) \right),$$

where $c(\{u\}) := \sum_{e \in E(T): u \in e} c(e)$. In this case, $\varrho_{\vec{u}u} = c(\{\vec{u}, \vec{\vec{u}}\})/c(\{\vec{u}, u\})$.

To define the potential V_T of the RWRE on T, we demand its increment between u and $\vec{u}$ to be given by $\log \varrho_{\vec{u}u}$, or in other words:

$$V_T(u) - V_T(\varrho) := \sum_{v \in [\varrho, u]]} \log \varrho_{\vec{v}v},$$

which is well-defined, up to a constant, for every node of T except the root and any of its children. It will be convenient to work with a slight modification of the trees under consideration. We add a nex vertex which we call the base and stick it to the root by a new edge with unit conductance, i.e. $c(\{\vec{\varrho}, \varrho\}) := 1$. This yields a planted tree $\bar{T}$. To keep our notation simple, even if the statements are expressed in terms of the planted tree $\bar{T}$, we still phrase them in terms of T. Setting $V_T(\varrho) := 0$ extends the definition of the potential to the whole vertex set of T. Now, observing that the potential is given pointwise at $u \in T \setminus \{\varrho\}$ by the telescopic sum

$$V_T(u) = \sum_{v\in[\varrho,u]]} \log \varrho_{\vec{v}v} = \sum_{v\in[\varrho,u]]} \Big[\log c(\{\vec{v},\vec{\vec{v}}\}) - \log c(\{\vec{v},v\})\Big] = \log c(\{\vec{u},u\})^{-1},$$

we deduce that the exponential of the potential at u is equal to the resistance $r(\{\vec{u},u\}) := c(\{\vec{u},u\})^{-1}$. Therefore, we can now define the potential as

$$V_T(u) = \begin{cases} \log r(\{\vec{u},u\}), & u \in T\setminus\{\varrho\}, \\ 0, & u = \varrho. \end{cases} \tag{5.2}$$

One of the crucial facts for the RWRE on tree-like spaces is that, for fixed ω, the random walk is a reversible Markov chain, and thus it was of no loss of generality to assume that the marginals of ω are defined as the transition probabilities of a weighted random walk on T, see [81, Section 9.1]. The RWRE on T, for fixed ω, can be described as an electrical network with resistances given by $r(\{\vec{u},u\}) = e^{V_T(u)}$, $u \in T$, and resistance metric

$$r(u_1,u_2) := \sum_{u\in[u_1,u_2]]} r(\{\vec{u},u\}) = \sum_{u\in[u_1,u_2]]} e^{V_T(u)}, \qquad u_1,u_2\in T, \tag{5.3}$$

with the convention of a sum taken over the empty set being equal to zero. The stationary measure of the RWRE on T, for fixed ω, is given by

$$\nu(\{u\}) := c(\{u\}) = \sum_{i=0}^{\xi(u)} r(\{u,u_i\})^{-1} = e^{-V_T(u)} + \sum_{i=1}^{\xi(u)} e^{-V_T(u_i)}, \qquad u\in T. \tag{5.4}$$

The reversibility means that, for all $u\in T$ and $0\le i\le \xi(u)$, we have

$$\begin{aligned}\nu(\{u\})\omega_{uu_i} = c(\{u\})\frac{c(\{u,u_i\})}{c(\{u\})} &= c(\{u,u_i\}) \\ &= c(\{u_i,u\}) = c(\{u_i\})\frac{c(\{u_i,u\})}{c(\{u_i\})} = \nu(\{u_i\})\omega_{u_iu}.\end{aligned}$$

If (T,r) is a metric tree, we denote by $\mathcal{C}(T)$ the space of continuous functions $f : T\to\mathbb{R}$ and by $\mathcal{C}_\infty$ the subspace of functions that are vanishing at infinity.

A continuous function is called locally absolutely continuous if for every $\varepsilon > 0$ and all subsets $T' \subseteq T$ with $\lambda(T') < \infty$ (recall the notion of the length measure λ introduced in Definition 2.1.2), there exists a $\delta \equiv \delta(T', \varepsilon)$, such that if $[[u_i, v_i]]_{i=1}^n \subseteq T'$ is a disjoint collection of arcs with $\sum_{i=1}^n r(u_i, v_i) < \delta$, then $\sum_{i=1}^n |f(u_i) - f(v_i)| < \varepsilon$. Denote the subspace of locally absolutely continuous functions by $\mathcal{A}$. Notice that in the case when (T, r) is a discrete metric tree $\mathcal{A}$ is equal to the space of continuous functions.

Consider the bilinear form

$$\mathcal{E}(f, g) := \frac{1}{2} \int \mathrm{d}\lambda \nabla f \cdot \nabla g \tag{5.5}$$

and its domain

$$\mathcal{D}(\mathcal{E}) := L^2(\nu) \cap \mathcal{C}_\infty \cap \{f \in \mathcal{A} : \nabla f \in L^2(\lambda)\}, \tag{5.6}$$

where the gradient, ∇f, of $f \in \mathcal{A}$ is the function, which is unique up to λ-null sets, that satisfies

$$\int_{u_1}^{u_2} \nabla f(u) \lambda(\mathrm{d}u) = f(u_2) - f(u_1), \qquad \forall u_1, u_2 \in T. \tag{5.7}$$

For its existence and uniqueness, see [12, Proposition 1.1]. The gradient, ∇f, of $f \in \mathcal{A}$ depends on the choice of the root ϱ, although, the bilinear form in (5.5) is independent of that choice, see [12, Remark 1.3].

Definition 5.1.1 (ν-symmetric Markov process)**.** *We call a Markov process X on $(T, \mathcal{B}(T))$ ν-symmetric if the transition function $\{T_t\}_{t>0}$ of X is ν-symmetric on $(T, \mathcal{B}(T))$ in the following sense:*

$$\int_T f(u)(T_t g)(u)\nu(\mathrm{d}u) = \int_T (T_t f)(u) g(u)\nu(\mathrm{d}u)$$

for any non-negative measurable functions f and g.

Theorem 5.1.1 ([12], [13])**.** *There exists a unique ν-symmetric strong Markov process $((X_t)_{t\geq 0}, \mathbf{P}^u, u \in T)$ associated with the regular Dirichlet form $(\mathcal{E}, \bar{\mathcal{D}}(\mathcal{E}))$ on the metric measure tree (T, r, ν), which is called the ν-speed motion on (T, r).*

If (T, r) is a compact real tree, then the ν-speed motion on (T, r) coincides with the ν-Brownian motion on T [12], i.e. a ν-symmetric strong Markov process with the following properties:

i) Continuous sample paths.

ii) Reversible with respect to the invariant measure ν.

iii) For every $u_1, u_2 \in T$ with $u_1 \neq u_2$,

$$\mathbf{P}^{u_3}(\tau_{u_1} < \tau_{u_2}) = \frac{r(u(u_1, u_2, u_3), u_2)}{r(u_1, u_2)}, \qquad u_3 \in T,$$

where $\tau_u := \inf\{t > 0 : X_t = u\}$ is the hitting time of $u \in T$, and $u(u_1, u_2, u_3)$ is the unique branch point of u_1, u_2 and u_3 in T.

iv) For $u_1, u_2 \in T$, the mean occupation measure for the process started at u_1 and killed upon hitting u_2 has density $2r(u(u_1, u_2, u_3), u_2)\nu(\mathrm{d}u_3)$, so that

$$\mathbf{E}^{u_1}\left(\int_0^{\tau_{u_2}} f(X_s)ds\right) = 2\int_T f(u_3)r(u(u_1, u_2, u_3), u_2)\nu(\mathrm{d}u_3),$$

for every $f \in \mathcal{C}(T)$.

If (T, r) is a discrete metric measure tree, then the ν-speed motion on (T, r) is the continuous-time nearest neighbor random walk on (T, r) with the following jump rates:

$$q(u_1, u_2)^{-1} := 2 \cdot \nu(\{u_1\}) \cdot r(u_1, u_2), \qquad u_1 \sim u_2. \tag{5.8}$$

Equivalently, the ν-speed motion on (T, r) is the continuous-time nearest neighbor random walk on (T, r) with associated Dirichlet form $(\mathcal{E}, \bar{\mathcal{D}}(\mathcal{E}))$:

$$\mathcal{E}(f, g) = (-Lf, g)_\nu, \tag{5.9}$$

where

$$Lf = \frac{1}{2\nu(\{u_1\})}\sum_{u_2 \sim u_1}\frac{1}{r(u_1, u_2)}(f(u_2) - f(u_1))$$

is the generator of the process, acting on continuous functions $f \in \mathcal{C}(T)$ that depend only on finitely many points of T.

Let (T, r, ν) be a compact real metric measure tree. To formalize the notion of the potential of diffusions on (T, r), which are not necessarily on natural scale, assume that we are further given a measure μ which is absolutely continuous with respect to the length measure λ and its density is given by

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(u) = e^{\phi(u)}, \tag{5.10}$$

where $\phi : T \to \mathbb{R}$ is a continuous function. For every $u_1, u_2 \in T$, let $r_\phi : T \times T \to \mathbb{R}_+$ defined by

$$r_\phi(u_1, u_2) := \int_{[[u_1,u_2]]} e^{\phi(u)} \lambda(\mathrm{d}u). \tag{5.11}$$

To justify the term potential on T given to ϕ, cf. (5.3). It is easy to check that r_ϕ defines a metric on T. In addition, r and r_ϕ are topologically equivalent and the metric space (T, r_ϕ) is also a compact real tree. Moreover, $(\mathcal{E}, \bar{\mathcal{D}}(\mathcal{E}))$ (see (5.5) and (5.6) with the difference that in (5.5) we integrate with respect to μ instead of λ) is a regular Dirichlet form. In this case we refer to the corresponding diffusion as the (ν, μ)-Brownian motion. The ν-Brownian motion on (T, r_ϕ) is equal in law with the (ν, μ)-Brownian motion on (T, r), see [12, Example 8.3]. In fact, for the previous statement to hold, ϕ needs not to be assumed continuous insofar as it has enough regularity for the integral in (5.11) to make sense and (T, r_ϕ) to be a locally compact real tree.

Now, we are ready to state our main assumption that corresponds to a metric measure version of Sinaĭ's model, that is when the potential converges to a Brownian motion. The natural tree-distance and the counting measure on the tree are replaced by the distorted resistance metric and the invariant measure of the RWRE on the tree, which are explicitly associated with the potential on the tree.

Assumption 5. *For a sequence $(\mathcal{T}_n, V_n)_{n \geq 1} \in \tilde{\mathbb{K}}$ of random elements built on a probability space with probability measure $\mathbf{P}$, where $\mathcal{T}_n := ((T_n, r_n, \varrho^n), \nu_n)$, $n \geq 1$ is a (locally finite) rooted plane metric measure tree with metric r_n as in (5.3), boundedly finite measure ν_n as in (5.4), and $V_n : T_n \to \mathbb{R}$ is the potential of the RWRE as defined in (5.2), we suppose that*

$$(T_n, V_n) \xrightarrow{(d)} (\mathcal{T}, \phi) \tag{5.12}$$

in the spatial Gromov-Hausdorff-vague topology, where $\mathcal{T} := ((T, r_\phi, \varrho), \nu_\phi)$ is a rooted real measure tree with metric r_ϕ as in (5.11), boundedly finite measure ν_ϕ, *and $\phi : T \to \mathbb{R}$ is a continuous potential on T as defined in (5.10). Moreover, suppose that the following non-explosion condition of the metrics is satisfied:*

$$\lim_{R\to\infty} \liminf_{n\to\infty} \mathbf{P}\left(r_n(\varrho^n, B_n(\varrho^n, R)^c) \geq \lambda\right) = 1, \qquad \forall \lambda \geq 0. \tag{5.13}$$

With their role as the scale function and the speed measure, r_n and ν_n will dictate the scaling of the RWRE. If Assumption 5 holds, as a corollary of [36, Theorem 1.2], it is possible to isometrically embed (T_n, r_n), $n \geq 1$ and (T, r_ϕ) into a common metric space (Z, d_Z) in such a way that the ν_n-speed motion on (T_n, r_n) converges weakly on $D(\mathbb{R}_+, Z)$ to the ν_ϕ-Brownian motion on (T, r_ϕ). Note that, r_n is a resistance metric associated with the bilinear form (5.9) and r_ϕ is a resistance metric associated with the bilinear form (5.5), when integrating with respect to μ instead of λ.

Theorem 5.1.2 (cf. **Croydon** [36]). *Let $(X_t^n)_{t\geq 0}$ be the random walk associated with a random environment $\omega(n)$, $n \geq 1$. Under Assumption 5, there exists a common metric space (Z, d_Z) onto which we can isometrically embed (T_n, r_n), $n \geq 1$ and (T, r_ϕ), such that*

$$\mathbb{P}^{\varrho^n}\left((X_t^n)_{t\geq 0} \in \cdot\right) \to \mathbb{P}^{\varrho}\left((X_t)_{t\geq 0} \in \cdot\right),$$

weakly on $D(\mathbb{R}_+, Z)$, where $(X_t)_{t\geq 0}$ is the ν_ϕ-Brownian motion on (T, r_ϕ). Here, $\mathbb{P}^{\varrho^n}$ and $\mathbb{P}^{\varrho}$ represent the annealed laws of the corresponding processes, obtained by integrating out the randomness of the elements of $\tilde{\mathbb{K}}$ with respect to $\mathbf{P}$.

Remark (cf. Croydon [36]). *When $(\mathcal{T}_n, (V_n, \psi_n))$, $n \geq 1$ and $(\mathcal{T}, (\phi, \psi))$ are random elements of $\tilde{\mathbb{K}}$, built on a probability space with probability measure $\mathbf{P}$, where ψ_n and ψ are continuous embeddings of (T_n, r_n), $n \geq 1$ and (T, r_ϕ) respectively, into a complete and separable metric space (K, d_K), Assumption 5 (with the probabilistic non-explosion of (5.13)) and its validity implies the annealed convergence of the embedded stochastic processes involved in Theorem 5.1.2:*

$$\mathbb{P}^{\varrho^n}\left((\psi_n(X_t^n))_{t\geq 0} \in \cdot\right) \to \mathbb{P}^{\varrho}\left((\psi(X_t))_{t\geq 0} \in \cdot\right),$$

weakly on $D(\mathbb{R}_+, K)$, where $\mathbb{P}^{\varrho^n}$ and $\mathbb{P}^{\varrho}$ represent the annealed laws of the corresponding processes, obtained by integrating out the randomness of the elements of $\tilde{\mathbb{K}}$ with respect to $\mathbf{P}$.

5.2 Convergence of Sinaĭ's random walk to the Brox diffusion

We introduce the one-dimensional RWRE considered early in the works of [97] and [98] (see also [57] and [69]) and studied extensively subsequently by many authors (we refer to [104] for a detailed account). Given a sequence $\omega = (\omega_z^-)_{z\in\mathbb{Z}}$ of i.i.d. random variables taking values in (0,1) and defined on a probability space $(\Omega, \mathcal{F}, P)$, the one-dimensional RWRE is the Markov chain $X = ((X_n)_{n\geq 1}, \mathbf{P}_\omega^u, u \in \mathbb{Z})$ that given ω has transition probabilities:

$$\mathbf{P}_\omega(X_{n+1} = z-1 | X_n = z) = \omega_z^-, \qquad \mathbf{P}_\omega(X_{n+1} = z+1 | X_n = z) = \omega_z^+ := 1 - \omega_z^-.$$

Let $\varrho_z := \omega_z^- / \omega_z^+$, $z \in \mathbb{Z}$ and assume that

$$E_P(\log \varrho_0) = 0, \qquad \sigma := \mathrm{Var}(\log \varrho_0) > 0, \tag{5.14}$$

$$P(\varepsilon \leq \omega_0^- \leq 1-\varepsilon) = 1, \text{ for some } \varepsilon \in (0, 1/2). \tag{5.15}$$

The first assumption ensures that the one-dimensional RWRE is recurrent, P-a.s. ω, while the second forces the environment to be non-deterministic. The last assumption, called uniform ellipticity, is usually used in the context of RWRE for technical reasons. Sinaĭ [97] showed that there exists a non-trivial random variable $b_1 : \Omega \to \mathbb{R}$, whose law was characterized later independently by Golosov [57] and Kesten [69], such that for any $\eta > 0$,

$$\mathbb{P}^u\left(\left| \frac{\sigma^2 X_n}{(\log n)^2} - b_1(\omega) \right| > \eta \right) \to 0, \tag{5.16}$$

as $n \to \infty$, where $\mathbb{P}^u$ is the annealed law of X defined as $\mathbb{P}^u(G) := \int \mathbf{P}_\omega^u(G) P(d\omega)$, for any fixed Borel set $G \subseteq \mathbb{Z}^{\mathbb{N}}$. This result was a consequence of a localization phenomenon that occurs, trapping the random walk in some valleys of its potential.

Brox [28] considered a one-dimensional diffusion process in a random Brownian environment W that formally solves the stochastic differential equation

$$dX_t = dB_t - \frac{1}{2} W'(X_t) dt, \qquad X_0 = 0, \tag{5.17}$$

where $(B_t)_{t\geq 0}$, $(W_1(x))_{x\geq 0}$, $(W_2(x))_{x\leq 0}$ are three mutually independent standard

Brownian motions, such that

$$W(x) := \begin{cases} \sigma W_1(|x|), & x \geq 0, \\ \sigma W_2(|x|), & x \leq 0, \end{cases} \tag{5.18}$$

for some $\sigma > 0$. Rigorously speaking we are considering a Feller-diffusion process X_t on $\mathbb{R}$ with the generator of Feller's canonical form

$$\frac{1}{2e^{-W(x)}}\frac{d}{dx}\left(\frac{1}{e^{W(x)}}\frac{d}{dx}\right).$$

Once one defines the conditioned process X_t given an environment W, using the law of total probability, one defines what the process X_t is.

Among those, he also showed that this real-valued stochastic process X_t converges very slowly, when $\sigma = 1$, to the same random variable b_1 as in (5.16). Namely, for every $\eta > 0$,

$$\mathbb{P}^u\left(\left|\alpha^{-2}X_{e^\alpha} - b_1(\omega)\right| > \eta\right) \to 0, \tag{5.19}$$

as $\alpha \to \infty$.

(5.16) and (5.19) show that the one-dimensional RWRE enjoys the same asymptotic properties as a one-dimensional diffusion process in a random Brownian environment, however this does not necessarily imply that Brox's diffusion is the continuous analogue of Sinaĭ's random walk. This question was answered in the affirmative by Seignourel [95] who proved the existence of a Donsker's invariance principle in a setting where one is allowed to parametrize the random environment appropriately at every step of the walk.

Theorem 5.2.1 (Seignourel [95]). *For every $m \geq 1$, consider a sequence of i.i.d. random variables $(\omega_z^-(m))_{z\in\mathbb{Z}}$, and for simplicity denote $\omega_z^-(1)$ by ω_z^-. Furthermore, suppose that (5.14) and (5.15) are satisfied, while also, for every $m \geq 1$ and for every $z \in \mathbb{Z}$,*

$$\omega_z^+(m) := 1 - \omega_z^-(m) \overset{(d)}{=} \left(1 + \varrho_z{}^{m^{-1/2}}\right)^{-1}, \tag{5.20}$$

which in other words means that, for every $m \geq 1$ and for every $z \in \mathbb{Z}$, $\varrho_z(m) := \omega_z^-(m)/\omega_z^+(m) \overset{(d)}{=} \varrho_z^{m^{-1/2}}$. If, for every $m \geq 1$, $(X_n^m)_{n\geq 1}$ denotes the random walk *associated with the random environment* $(\omega_z^-(m))_{z\in\mathbb{Z}}$*, then*

$$(m^{-1}X^m_{\lfloor m^2 t\rfloor})_{t\geq 0} \xrightarrow{(d)} (X_t)_{t\geq 0}$$

in distribution in $D([0,\infty))$*, where* $(X_t)_{t\geq 0}$ *is the Brox diffusion.*

We undertake the task of generalising the result for Seignourel's model by effectively removing the uniform ellipticity condition. Such a gesture is meaningful in that it allows us to include applications of this theorem to environments that are not uniformly elliptic, such as Dirichlet environments. A particular model of interest that famously falls into this class is the edge (linearly) reinforced random walk on locally finite directed graphs. For an overview on random walks in Dirichlet random environment (RWDE) we refer to [94].

In a second level the i.i.d. assumption made by Seignourel [95] is not essential as soon as we suppose that the potential of the random walk associated with the parametrized environment converges weakly to a two-sided Brownian motion. Recalling some basic definitions from Section 5, for every $m \geq 1$,

$$V_x^m := \begin{cases} \frac{1}{\sqrt{m}}\sum_{i=1}^{x}\log\varrho_i, & x\geq 1,\\ 0, & x=0,\\ -\frac{1}{\sqrt{m}}\sum_{i=x+1}^{0}\log\varrho_i, & x\leq -1. \end{cases}$$

is the potential of the one-dimensional RWRE changed at step m according to (5.20), and now we are ready to make our assumption precise. It clarifies why in order to get a Donsker's theorem in random medium one is forced to "flatten" the environment in the first place.

Assumption 6 (Sinaĭ's regime)**.** *Suppose that* $(V^m_{\lfloor mx\rfloor})_{x\in\mathbb{R}}$ *converges weakly to* $(W(x))_{x\in\mathbb{R}}$*, where* $(W(x))_{x\in\mathbb{R}}$ *is a two-sided Brownian motion as in* (5.18)*.*

By direct calculation it can be verified that, for fixed $\omega(m)$, $m\geq 1$, the RWRE $(X_n^m)_{n\geq 1}$, $m\geq 1$, is a reversible Markov chain and the stationary reversible measure which is unique up to multiplication by a constant is given by

$$\nu_{\omega(m)}(x) = \begin{cases} (1+\varrho_x(m))\left(\prod_{i=1}^{x}\varrho_i(m)\right)^{-1}, & x\geq 1,\\ 1+\varrho_0(m), & x=0,\\ (1+\varrho_x(m))\prod_{i=x+1}^{0}\varrho_i(m), & x\leq -1. \end{cases} \tag{5.21}$$

Here, the reversibility means that, for all $n \geq 0$ and $x, y \in \mathbb{Z}$, we have that

$$\nu_{\omega(m)}(x) P_{\omega(m)}(X_n^m = y | X_0^m = x) = \nu_{\omega(m)}(y) P_{\omega(m)}(X_n^m = x | X_0^m = y).$$

Sticking to the interpretation of the one-dimensional RWRE as an electrical network with resistances given by $r_{\omega(m)}(x-1, x) = e^{V_{x-1}^m}$, $x \in \mathbb{Z}$, we can rewrite (5.21) as

$$\nu_{\omega(m)}(x) = e^{-V_x^m} + e^{-V_{x-1}^m}, \qquad x \in \mathbb{Z}. \tag{5.22}$$

Moreover, we endow $\mathbb{Z}$ with the resistance metric $r_{\omega(m)} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}_+$ that satisfies $r_{\omega(m)}(x, x) := 0$, for every $x \in \mathbb{Z}$, and

$$r_{\omega(m)}(x, y) := \sum_{z=x}^{y-1} r_{\omega(m)}(z, z+1) = \sum_{z=x}^{y-1} e^{V_z^m}, \qquad x < y. \tag{5.23}$$

The one-dimensional lattice viewed as a rooted metric measure space endowed with the finite measure and the resistance metric defined in (5.22) and (5.23) respectively, in Sinaĭ's regime converges weakly in the spatial Gromov-Hausdorff-vague topology as indicated by the next theorem.

Theorem 5.2.2. *Under Assumption 6,*

$$\left((\mathbb{Z}, m^{-1} r_{\omega(m)}, 0), m^{-1} \nu_{\omega(m)}, V^m\right) \xrightarrow{(d)} ((\mathbb{R}, r, 0), \nu, W), \qquad m \to \infty,$$

in the spatial Gromov-Hausdorff-vague topology, where

$$r(x, y) := \int_{[x \wedge y, x \vee y]} e^{W(z)} dz, \tag{5.24}$$

for every $x, y \in \mathbb{R}$ and

$$\nu(A) := \int_A 2e^{-W(x)} dx, \tag{5.25}$$

for every $A \in \mathcal{B}(\mathbb{R})$.

Proof. By Skorohod's representation theorem, there exists a probability space on which the convergence

$$(V_{\lfloor mx \rfloor}^m)_{x \in \mathbb{R}} \xrightarrow{(d)} (W(x))_{x \in \mathbb{R}} \tag{5.26}$$

holds almost-surely with respect to the uniform norm on compact intervals. Define a correspondence R_m between $\mathbb{Z}$ and $\mathbb{R}$ by setting $(i, s) \in R_m$ if and only if

$i = \lfloor ms \rfloor$. We will bound the distortion of R_m. Suppose that $(i,s),(j,t) \in R_m$, such that $s \leq t$. Then,

$$|m^{-1}r_{\omega(m)}(i,j) - r(s,t)|$$

$$= \left| m^{-1} \sum_{z=i}^{j-1} e^{V_z^m} - \int_s^t e^{W(u)} du \right| = \left| \int_{\lfloor ms \rfloor/m}^{\lfloor mt \rfloor/m} e^{V_{\lfloor mu \rfloor}^m} du - \int_s^t e^{W(u)} du \right|, \qquad (5.27)$$

whichn turn, using the triangle inequality, can be bounded above by

$$\left| \int_s^t e^{V_{\lfloor mu \rfloor}^m} du - \int_s^t e^{W(u)} du \right| + \left| \int_{\lfloor ms \rfloor/m}^{\lfloor mt \rfloor/m} e^{V_{\lfloor mu \rfloor}^m} du - \int_s^t e^{V_{\lfloor mu \rfloor}^m} du \right|$$

$$\leq \int_s^t |e^{V_{\lfloor mu \rfloor}^m} - e^{W(u)}| du + \int_{\lfloor ms \rfloor/m}^{s} |e^{V_{\lfloor mu \rfloor}^m}| du + \int_{\lfloor mt \rfloor/m}^{t} |e^{V_{\lfloor mu \rfloor}^m}| du. \qquad (5.28)$$

Then, $\mathrm{dis}(R_m)$ converges to 0 uniformly in $s,t \in [-R,R]$, for some $R > 0$, see (5.27) and (5.28), which combined give us the following:

$$\mathrm{dis}(R_m) = \sup\{|m^{-1}r_{\omega(m)}(i,j) - r(s,t)| : (i,s),(j,t) \in R_m\}$$

$$\leq 2R||e^{V_{\lfloor m\cdot \rfloor}^m} - e^W||_{\infty,[-R,R]} + 2m^{-1}||e^{V_{\lfloor m\cdot \rfloor}^m}||_{\infty,[-2R,2R]} \xrightarrow[m\to\infty]{} 0. \qquad (5.29)$$

Recall that $m^{-1}\nu_{\omega(m)}$ puts mass $m^{-1}(e^{-V_i^m} + e^{-V_{i-1}^m})$ on $i \in \mathbb{Z}$. Then, we may couple $m^{-1}\nu_{\omega(m)}$ and ν by taking $U \sim U[-R,R]$ and taking π to be the law of the pair

$$(\lfloor mU \rfloor, 2e^{-W(U)}).$$

This is precisely the natural coupling π induced by the correspondence R_m. Therefore, $\pi(R_m^c) = 0$. Since, for every $R \geq 0$,

$$d_{\tilde{\mathbb{K}}}\left(\left(\left(\mathbb{Z}, m^{-1}r_{\omega(m)}, 0\right)\big|_R, m^{-1}\nu_{\omega(m)}\big|_R, V^m\big|_R\right), \left(\left(\mathbb{R}, r, 0\right)|_R, \nu|_R, W|_R\right)\right)$$

$$\leq \frac{1}{2}\mathrm{dis}(R_m) + \pi(R_m^c) + ||V_{\lfloor m\cdot \rfloor}^m - W||_{\infty,[-R,R]}$$

the result follows by (5.29) and the convergence in (5.26), which holds almost-surely with respect to the uniform norm on $[-R, R]$.

$\square$

Let $R > 0$. It is obvious that

$$\liminf_{m\to\infty} r_{\omega(m)}(0, B_{\omega(m)}(0, R)^c) \geq \frac{R}{2},$$

and therefore taking the limit as $R \to \infty$ yields that (5.13) is satisfied. Combining this with Theorem 5.2.2 allows us to deduce that Assumption 5 is fulfilled. Thus, as a consequence of Theorem 5.1.2, the $\nu_{\omega(m)}$-speed motion on $(\mathbb{Z}, r_{\omega(m)}, 0)$ converges weakly in $D([0, \infty))$ to the ν-speed motion on $(\mathbb{R}, r, 0)$. The $\nu_{\omega(m)}$-speed motion on $(\mathbb{Z}, r_{\omega(m)})$ is the continuous-time nearest neighbor random walk on $(\mathbb{Z}, r_{\omega(m)})$ with jumps rescaled by m^{-1} and time speeded up by

$$\nu_{\omega(m)}(x)^{-1}(r_{\omega(m)}(x, x+1)^{-1} + r_{\omega(m)}(x-1, x)^{-1}) = m^2, \qquad x \in \mathbb{Z},$$

which, is equal in law to $(m^{-1}X^m_{\lfloor m^2 t\rfloor})_{t\geq 0}$.

It remains to identify (in law) the ν-speed motion on $(\mathbb{R}, r, 0)$ with the Brox model, see (5.17). Fixing the environment W, $(X_t)_{t\geq 0}$ is a Feller-diffusion on $\mathbb{R}$ having infinitesimal generator of Feller's canonical form

$$\frac{1}{2e^{-W(x)}}\frac{d}{dx}\left(\frac{1}{e^{W(x)}}\frac{d}{dx}\right).$$

In other words, $(X_t)_{t\geq 0}$ is a diffusion on $\mathbb{R}$ with differentiable scale function

$$s(x) := \int_0^x e^{W(z)}dz, \qquad x \in \mathbb{R},$$

and speed measure

$$\nu(A) := \int_A 2e^{-W(x)}dx, \qquad A \in \mathcal{B}(\mathbb{R}),$$

which is the same as the one in (5.25). Then, X is the continuous strong Markov process associated with the Dirichlet form

$$\mathcal{E}(f, g) := \frac{1}{2}\int \frac{dz}{s'(z)} f'(z) \cdot g'(z),$$

for every $f, g \in L^2(\nu) \cap \mathcal{C}_\infty \cap \mathcal{A}$, such that $\mathcal{E}(s, g) < \infty$, where here $\mathcal{A}$ is the space of absolutely continuous functions. Note that, for all $x, y \in \mathbb{R}$,

$$r(x, y) = \int_{[x \wedge y, x \vee y]} s'(z) dz,$$

which can be seen to induce that $(\mathbb{R}, r, 0)$ is a locally compact real tree with length measure $s'(z)dz$. The gradient $\nabla_r f$, of $f \in \mathcal{A}$ is the function, which is unique up to $s'(z)dz$-zero sets, that satisfies

$$\int_x^y \nabla_r f(z) s'(z) dz = f(y) - f(y),$$

for every $x, y \in \mathbb{R}$, see (5.7). Therefore, $\nabla_r f = f'/s'$. Using this information, by the following calculation, we find that

$$\begin{aligned}
\mathcal{E}(f, g) &= \frac{1}{2} \int \frac{dz}{s'(z)} f'(z) \cdot g'(z) dz \\
&= \frac{1}{2} \int \frac{dz}{s'(z)} (\nabla_r f(z) s'(z)) \cdot (\nabla_r g(z) s'(z)) = \frac{1}{2} \int s' dz \nabla_r f \cdot \nabla_r g,
\end{aligned}$$

for every $f, g \in L^2(\nu) \cap \mathcal{C}_\infty \cap \mathcal{A}$, such that $\mathcal{E}(f, g) < \infty$. To conclude, the ν-speed motion on $(\mathbb{R}, r, 0)$ is equal in law with X. We have thus successfully proven Seingourel's result to hold for a wider class of random walks in random environment.

Theorem 5.2.3. *Let, for every $m \geq 1$, $(X_n^m)_{n \geq 1}$ denote the random walk associated with the random environment under which Assumption 6 holds. Then,*

$$(m^{-1} X^m_{\lfloor m^2 t \rfloor})_{t \geq 0} \xrightarrow{(d)} (X_t)_{t \geq 0}$$

in distribution in $D([0, \infty))$, where $(X_t)_{t \geq 0}$ is the Brox diffusion.

5.3 Convergence of a random walk with barriers

A model with infinitely many barriers was considered by Carmona in [30] in order to study the large time behavior of the solution of (5.17) when the random coefficient W' is replaced by the formal derivative of a spatial Lévy process. The random environment consists of a sequence of barriers $(\tau_z)_{z \in \mathbb{Z}}$ such that their increments $(\tau_z - \tau_{z-1})_{z\in\mathbb{Z}}$ form a sequence of independent geometric random variables of parameter $\alpha \in (0,1)$. To construct the random environment rigorously consider a sequence of Bernoulli random variables $(\xi_z)_{z\in\mathbb{Z}}$ of parameter $\alpha \in (0,1)$, i.e. $P(\xi_1 = 1) = 1 - P(\xi_1 = 0) = \alpha$ and let

$$\beta_\alpha(z) := \begin{cases} \sum_{k=1}^{z} \xi_k, & z \geq 1, \\ 0, & z = 0, \\ -\sum_{k=z}^{-1} \xi_k, & z \leq -1. \end{cases} \tag{5.30}$$

Then, setting $\tau_z := \inf\{r \in \mathbb{Z} : \beta_\alpha(r) = z\}$ yields the desired property for the increments of $(\tau_z)_{z\in\mathbb{Z}}$. The random walk in the random environment τ is introduced as a simple random walk away from the level of the set $\{\tau_z : z \in \mathbb{Z}\}$. When it reaches one of the barriers a biased coin is tossed, with probability of heads thrown being $p \in (0,1)$, it chooses to move to the right with probability p or otherwise to the left with probability $q := 1 - p$. In other words, the random walk in the random environment τ is the Markov chain $((X_n)_{n\geq 1}, \mathbf{P}_\tau^u, u \in \mathbb{Z})$ that given τ has transition probabilities:

$$\begin{aligned} 1 - P_\tau(X_{n+1} = z-1 | X_n = z) &= P_\tau(X_{n+1} = z+1 | X_n = z) \\ &= \begin{cases} \frac{1}{2}, & z \notin \{\tau_z : z \in \mathbb{Z}\}, \\ p, & z \in \{\tau_z : z \in \mathbb{Z}\}. \end{cases} \end{aligned}$$

To treat this example as part of the framework in which Assumption 6 was imposed we need to generalize the Gromov-Hausdorff-vague topology on rooted metric measure spaces endowed with a càdlàg function $\phi : \mathbb{R} \to \mathbb{R}$. To do this we replace $d_E(\phi(z), \phi'(z'))$ that appears in the definition of the metric on $\tilde{\mathbb{K}}$ with $d_{J_1}(\phi(z), \phi'(z'))$, where d_{J_1} denotes the Skorohod metric on $D(\mathbb{R})$. It can be checked that $\tilde{\mathbb{K}}$ with this new metric constitutes a separable metric space, see Section 2.3. In the light of this consideration we can reformulate Assumption 6 to include one-dimensional diffusions with jumps. Namely, suppose that the limiting process $(W(x))_{x\in\mathbb{R}}$ in Assumption 6 is a two-sided Lévy process and that the convergence in distribution takes place on $D(\mathbb{R})$.

To write down the potential first observe that $\varrho_z = 1$ if and only if $z \notin \{\tau_z : z \in \mathbb{Z}\}$. Therefore, observing that the set of barriers $\{\tau_z : z \in \mathbb{Z}\}$ is a.s. identical to the set $\{z \in \mathbb{Z} : \xi_z = 1\}$, we have that

$$V_z = \sum_{k=1}^{z} \log \varrho_z = \log\left(\frac{q}{p}\right) \sum_{k=1}^{z} \xi_k = \log\left(\frac{q}{p}\right) \beta_\alpha(z), \qquad z \geq 1.$$

Repeating the same calculation for $z \leq -1$ implies that $V_z = \log(q/p)\beta_\alpha(z)$, for every $z \in \mathbb{Z}$.

To obtain in the limit a general Lévy process, and consequently a Brownian motion in random Lévy potential as the scaling limit of the random walk with infinitely many barriers, we normalize the random media appropriately. Let $\lambda > 0$, and for every $n \geq \lambda$ consider the normalized environment $(\beta^n_{\lambda/n}(z))_{z\in\mathbb{Z}}$ defined as in (5.30), where this time the Bernoulli trials have probability of success equal to λ/n. To verify that this is indeed the correct choice, check that the following conditions are satisfied.

$$\sum_{k=1}^{\lfloor nx \rfloor} P(\xi_k = 1) = \lambda \cdot \frac{\lfloor nx \rfloor}{n} \to \lambda x \in (0, \infty), \qquad \max_{1 \leq k \leq \lfloor nx \rfloor} P(\xi_k = 1) = \frac{\lambda}{n} \to 0,$$

for every $x > 0$. These are sufficient, see [50, Theorem 3.6.1] to allow us to deduce from the weak law of small numbers that, for fixed $x > 0$, $\beta^n_{\lambda/n}(\lfloor nx \rfloor)$ converges weakly to a Poisson random variable with mean λx. For an alternative proof of this fact using characteristic functions, see [50, Appendix B]. Therefore, for the two-sided process $(V^n_{\lfloor nx \rfloor})_{x\in\mathbb{R}}$ that has independent increments, we have that

$$(V^n_{\lfloor nx \rfloor})_{x\in\mathbb{R}} \xrightarrow{(d)} \log\left(\frac{q}{p}\right) (N(x))_{x\in\mathbb{R}}, \tag{5.31}$$

weakly on $D(\mathbb{R})$, where $(N(x))_{x\in\mathbb{R}}$ is a Poisson process with parameter $\lambda > 0$. Consequently, since the proof of Theorem 5.2.2 remains unchanged,

$$\left(\left(\mathbb{Z}, n^{-1} r_{\tau^n}, 0\right), n^{-1}\nu_{\tau^n}, V^n\right) \xrightarrow{(d)} \left((\mathbb{R}, r, 0), \nu, \log(q/p)N\right), \qquad n \to \infty, \tag{5.32}$$

in the spatial Gromov-Hausforff-vague topology, where $\tau^n_z := \inf\{r \in \mathbb{Z} : \beta^n_{\lambda/n}(r) = z\}$, see (5.22) and (5.23) for a definition of ν_{τ^n} and r_{τ^n} respectively. Slightly abusing notation, r and ν stand for (5.24) and (5.25) with W replaced by $\log(q/p)N$. The following result, that was conjectured by Carmona [30] and originally proved by Seignourel [95], is deduced by using (5.31), (5.32) and following the proof of

Theorem 5.2.3.

Theorem 5.3.1. *Let $\lambda > 0$, and for every $n \geq \lambda$ consider the random walk $(X_m^n)_{m\geq 1}$ associated with the random environment τ^n. Then,*

$$(n^{-1} X^n_{\lfloor n^2 t \rfloor})_{t\geq 0} \xrightarrow{(d)} (X_t)_{t\geq 0},$$

weakly on $D([0,\infty))$, where $(X_t)_{t\geq 0}$ is a solution to the SDE

$$dX_t = dB_t - \frac{1}{2}\left(\log\left(\frac{q}{p}\right) N'(X_t)\right) dt, \qquad X_0 = 0,$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion independent of N.

Remark. *One way to see that the process X_t exists is by noticing that its generator would take the form*

$$\frac{1}{2e^{-\log(\frac{q}{p})N(x)}} \frac{d}{dx}\left(\frac{1}{e^{\log(\frac{q}{p})N(x)}} \frac{d}{dx}\right).$$

Once one defines the conditioned process X_t given the environment N, using the law of total probability, one defines what the process X_t really is.

5.4 Random walk on the range of a branching random walk

We can define biased random walks on graphs generated by conditioned branching random walks. For a rooted finite ordered tree T with root ϱ, in which every edge e is marked by a real-valued vector $y(e)$, given a value function $y : E(T) \to \mathbb{R}^d$, we define a map $\phi : T \to \mathbb{R}^d$ by setting $\phi(\varrho) := 0$, $\phi(\vec{\varrho}) := 0$ and

$$\phi(u) := \sum_{e\in E_{\varrho,u}} y(e), \qquad u \in T \setminus \{\varrho\}, \tag{5.33}$$

where the sum is taken over the set of all edges contained in the unique path between ϱ and u. Also, we interpolate linearly along the edges. Let $\{(T_n, \phi_n)\}_{n\geq 1}$ be a family of random spatial graph trees, where T_n is generated by a Galton-Watson process with critical offspring distribution ξ conditioned to have total progeny

n. In addition, we demand ξ to have finite variance $\sigma_\xi^2 < \infty$ and exponential moments, i.e. $\mathbf{E}(e^{\lambda\xi}) < \infty$, for some $\lambda > 0$. Conditional on T_n, the increments $(y(e))_{e \in E(T_n)}$ of the spatial element ϕ_n are independent and identically distributed as a mean 0 random variable Y with finite variance $\Sigma_Y^2 < \infty$ (Σ_Y is a positive definite $d \times d$-matrix) that furthermore satisfies the tail condition:

$$\mathbf{P}(d_E(0,Y) \geq y) = o(y^{-4}),$$

where d_E denotes the usual Euclidean metric in $\mathbb{R}^d$. Given the other assumptions that we are making, [64, Theorem 2] ensures that the fourth order polynomial tail decay is necessary to obtain the convergence of the tours of T_n, i.e. the two-dimensional process $(C_n(i), R_n(i))$ supported on $\{0, ..., 2n\}$, such that the contour function $C_n(i)$ traces the distance to the root of the position of a particle that visits the outline of T_n from left to right at unit speed, and the head function $R_n(i) := \phi_n(u_i^n)$, if u_i^n denotes the i-th visited vertex in the contour exploration of T_n, keeps record of the points of the branching random walk ϕ_n. Note that, C_n determines the skeleton of the tree and R_n, via its increments, all the values.

Hence, for each $u \in T_n$ and conditional on T_n, $\phi_n(u)$ is a simple random walk on $\mathbb{R}^d$ with i.i.d. increments distributed as Y and number of steps given by the depth of the path from the root ϱ^n to u. The random multiset of trajectories is called a branching random walk. Let $\mathcal{G}_n = (V(\mathcal{G}_n), E(\mathcal{G}_n))$ be the graph with vertex set

$$V(\mathcal{G}_n) := \{x \in \mathbb{R}^d : x = \phi_n(u) \text{ with } u \in T_n\}$$

and edge set

$$E(\mathcal{G}_n) := \{\{x_1, x_2\} \in \mathbb{R}^d \times \mathbb{R}^d : x_i = \phi_n(u_i),\ i = 1,2 \text{ with } \{u_1, u_2\} \in E(T_n)\}.$$

Fix a parameter $\beta \geq 1$, and to each edge $\{x_1, x_2\} \in E(\mathcal{G}_n)$, assign the conductance

$$c(\{x_1, x_2\}) := \beta^{\max\{\phi_n^{(1)}(u_1), \phi_n^{(1)}(u_2)\}}$$

with $\{u_1, u_2\} \in E(T_n)$, where $\phi_n^{(1)}(u_i)$ denotes the first coordinate of $\phi_n(u_i)$, $i = 1, 2$. Observe that $c(\{\phi_n(\overleftarrow{\varrho^n}), \phi_n(\varrho^n)\}) = \beta^{\max\{\phi_n^{(1)}(\overleftarrow{\varrho^n}), \phi_n^{(1)}(\varrho^n)\}} = 1$, which is compatible with our convention of putting a unit conductance between the root and its base. The biased random walk on $\mathcal{G}_n$ is the Markov chain $X =$

$((X_n)_{n\geq 0}, \mathbf{P}^x_{\mathcal{G}_n}, x \in V(\mathcal{G}_n))$ on $V(\mathcal{G}_n)$ with transition probabilities given by

$$\mathbf{P}_{\mathcal{G}_n}(x_1, x_2) := \frac{c(\{x_1, x_2\})}{c(\{x_1\})},$$

where the normalization is defined by $c(\{x_1\}) := \sum_{e\in E(\mathcal{G}_n):x_1\in e} c(e)$. If $\beta > 1$, then the biased random walk X has a directional preference towards the first coordinate. On the other hand, if $\beta = 1$, there is no bias and we end up with the simple random walk on $\mathcal{G}_n$.

The RWRE on T_n is going to be of particular interest. Firstly, adopting the notation that was introduced in Section 5, the random environment at every vertex $u \in T_n$ will be represented by a random sequence $(\omega_{uu_i})_{i=0}^{\xi(u)}$ in $(0,1)^{\xi(u)}$ such that $\sum_{i=0}^{\xi(u)} \omega_{uu_i} = 1$. The RWRE on T_n will be the time-homogeneous Markov chain $X' = ((X'_n)_{n\geq 0}, \mathbf{P}^u_\omega, u \in T_n)$ taking values on T_n with transition probabilities given by (5.1). To connect this model with the biased random walk on the critical branching random walk conditioned to have n particles, suppose that the marginals of the environment are defined, for each $u \in T_n$, as follows:

$$(\omega_{uu_i})_{i=0}^{\xi(u)} = (\mathbf{P}_{\mathcal{G}_n}(\phi_n(u), \phi_n(u_i)))_{i=0}^{\xi(u)}.$$

For this choice of random environment, the quenched law of $\phi_n(X')$ is the same as that of X, and consequently the same holds for the corresponding annealed laws. This is immediate regarding the following relations:

$$(\mathbf{P}_{\mathcal{G}_n}(\phi_n(u), \phi_n(u_i)))_{i=0}^{\xi(u)} = \left(\frac{c(\{\phi_n(u), \phi_n(u_i)\})}{c(\{\phi_n(u)\})} : 0 \leq i \leq \xi(u)\right), \qquad u \in T_n.$$

To connect the first coordinate of the random embedding ϕ_n with the potential of the RWRE on T_n, let $(\Delta_n(u))_{u\in T_n}$ be its increments process, i.e.

$$\Delta_n(u) := \phi_n^{(1)}(u) - \phi_n^{(1)}(\vec{u}).$$

If the environment is defined as in the previous paragraph, $\log c(\{\phi_n(\vec{u}), \phi_n(u)\})^{-1} = -\log\beta \cdot \max\{\phi_n^{(1)}(\vec{u}), \phi_n^{(1)}(u)\}$. Therefore, the potential $(V_n(u))_{u\in T_n}$ of the random walk in a random environment on T_n, which is obtained by (5.2), satisfies

$$V_n(u) = -\log\beta\left(\phi_n^{(1)}(\vec{u}) + \max\{0, \Delta_n(u)\}\right), \qquad u \in T_n, \tag{5.34}$$

which demonstrates that if the individual increments are small, the potential of the RWRE on T_n is nearly given by a negative constant multiple of the first coordinate of ϕ_n.

We demonstrate that V_n, when rescaled, converges to an embedding of the Brownian CRT into the Euclidean space, so that an arc of length t in the Brownian CRT is mapped to the range of a Brownian motion run for time t. In other words, if $\mathcal{T}_e$ denotes the Brownian CRT , consider a tree-indexed Gaussian process $(\phi(\sigma))_{\sigma\in\mathcal{T}_e}$, built on a probability space with probability measure £$\mathbf{P}$, with $\mathbf{E}\phi(\sigma) = 0$, and $\mathrm{Cov}(\phi(\sigma), \phi(\sigma')) = d_e(\varrho, \sigma \wedge \sigma')I$, where I is the d-dimensional identity matrix. For almost-every realization of $\mathcal{T}_e$ (w.r.t the normalized Itô excursion measure $\mathbb{N}_1$), there exists a $\mathbf{P}$-a.s. continuous version of ϕ, see (51) in [49] for details. We keep the notation ϕ for this version.

For an underlying tree that satisfies the assumptions we made in the start of the section, [32, Corollary 10.3] ensures the following distributional convergence in $\tilde{\mathbb{K}}$. If d_{T_n} is the shortest path metric and μ_{T_n} is the uniform probability measure on the vertices of T_n, we have that

$$\left((T_n, n^{-1/2}d_{T_n}, \varrho^n), \mu_{T_n}, n^{-1/4}\phi_n\right) \xrightarrow{(d)} ((\mathcal{T}_e, \sigma_T d_e, \varrho), \mu_{\mathcal{T}_e}, \Sigma_\phi\phi)\,, \tag{5.35}$$

where $\sigma_T := \frac{2}{\sigma_\xi}$ and $\Sigma_\phi := \Sigma_Y\sqrt{\frac{2}{\sigma_\xi}}$. The limiting object $(\mathcal{T}_e, d_e)$ is a real tree coded by a normalized Brownian excursion $e := (e(t))_{0\leq t\leq 1}$, see (2.2) and (2.4). Combining (5.34) with (5.35) yields

$$((T_n, n^{-1/2}d_{T_n}, \varrho^n), \mu_{T_n}, n^{-1/4}\phi_n, n^{-1/4}V_n) \xrightarrow{(d)} ((\mathcal{T}_e, \sigma_T d_e, \varrho), \mu_{\mathcal{T}_e}, \Sigma_\phi\phi, \sigma_{\beta,\phi}\phi^{(1)}), \tag{5.36}$$

in the spatial Gromov-Hausdorff-vague topology, where $\phi^{(1)}$ denotes the first coordinate of ϕ and $\sigma_{\beta,\phi} = -\log\beta\cdot(\Sigma_\phi)_{11}$. It is natural to ask whether there is a certain regime in which the biased random walk on large critical branching random walk possesses a scaling limit. Answering the question posed above, (5.36) can be informative as it designates a discrete scheme in which the bias must be changed at every step. To be more precice, for every $n \geq 1$, let $(X^n_m)_{m\geq 1}$ denote the biased random walk on $\mathcal{G}_n$ with bias parameter $\beta_n := \beta^{n^{-1/4}}$, for some $\beta > 1$. We refer to this regime as the weakly biased regime on account of the "flattening" that the bias has to undergo. Observe that, for every $n \geq 1$, $(n^{-1/4}V_n(u))_{u\in T_n}$ is the potential of the RWRE on T_n changed at every step n according to

$$(c_n(\{x_1,x_2\}))_{\{x_1,x_2\}\in E(\mathcal{G}_n)} := \left(\beta^{n^{-1/4}\max\{\phi_n^{(1)}(u_1),\phi_n^{(1)}(u_2)\}}\right)_{\{u_1,u_2\}\in E(T_n)}.$$

Then, in conjunction with Section 5 and (5.4), for fixed environment, the stationary reversible measure of the weakly biased random walk $(X_m^n)_{m\geq 1}$ is unique up to multiplication by a constant and is given pointwise in u by

$$\nu_n(\{u\}) = e^{-n^{-1/4}V_n(u)} + \sum_{u'\sim u, u'\neq \overleftarrow{u}} e^{-n^{-1/4}V_n(u)}, \qquad u\in T_n, \tag{5.37}$$

where the sum is taken over the set of all vertices contained in the neighborhood of u excluding its parent. Moreover, the resistance metric with which T_n is endowed satisfies $r_n(u,u):=0$, for every $u\in T_n$, and

$$r_n(u_1,u_2) := \sum_{u\in[u_1,u_2]]} e^{n^{-1/4}V_n(u)}, \qquad u_1,u_2\in T_n \text{ with } u_1\neq u_2. \tag{5.38}$$

The rest of the section is devoted in verifying that the analogue of (5.36) indeed holds when the shortest path metric d_{T_n} and the uniform probability measure on the vertices of T_n are distorted by continuous functionals of the potential of the weakly biased random walk.

Theorem 5.4.1. *As $n\to\infty$,*

$$((T_n, n^{-1/2}r_n, \varrho^n), (2n)^{-1}\nu_n, n^{-1/4}\phi_n, n^{-1/4}V_n) \xrightarrow{(d)} ((\mathcal{T}_e, \sigma_T r_{\phi^{(1)}}, \varrho), \nu_{\phi^{(1)}}, \Sigma_\phi \phi, \sigma_{\beta,\phi}\phi^{(1)}),$$

in the spatial Gromov-Hausdorff-vague topology, where

$$r_{\phi^{(1)}}(u_1,u_2) := \int_{[[u_1,u_2]]} e^{\sigma_{\beta,\phi}\phi^{(1)}(v)}\lambda(\mathrm{d}v), \tag{5.39}$$

for every $u_1,u_2\in\mathcal{T}_e$ and $\nu_{\phi^{(1)}}$ is the mass measure defined as the image measure by the canonical projection $p_{\tilde{e}}$ of the Lebesgue measure on $[0,1]$, see (2.4), where

$$\tilde{e} := \left(\int_{[[p_e(0),p_e(t)]]} e^{-\sigma_{\beta,\phi}\phi^{(1)}(v)}\lambda(\mathrm{d}v) : 0\leq t\leq 1\right). \tag{5.40}$$

(note that $\tilde{e}:[0,1]\to\mathbb{R}_+$ is a (random) continuous function such that $\tilde{e}(0)=\tilde{e}(1)=0$, and therefore $p_{\tilde{e}}$ is well-defined).

Proof. Using Skorohod's representation theorem, we can assume that we are working on a probability space on which the distributional convergence of the normalized contour function of T_n,

$$(C_{(n)}(t))_{0\leq t\leq 1} := \left(\frac{C_n(2nt)}{\sqrt{n}} : 0 \leq t \leq 1\right),$$

to a normalized Brownian excursion $e := (e(t))_{0\leq t\leq 1}$, $C_{(n)} \xrightarrow{(d)} \sigma_T e$ in $C([0,1], \mathbb{R}_+)$ [6], holds in the almost-sure sense. We build a correspondence between T_n and $\mathcal{T}_e$ as follows. Let R_n be the image of the set (i,t) by the mapping $(i,t) \mapsto (u_i^n, p_e(t))$ from $\{0,...,2n\} \times [0,1]$ to $T_n \times \mathcal{T}_e$ such that $i = \lfloor 2nt \rfloor$, where u_i^n is the i-th visited vertex in the contour exploration of T_n and p_e denotes the canonical projection from $[0,1]$ to $\mathcal{T}_e$. Note that this correspondence also associates the root u_0^n of T_n with the root $p_e(0)$ of $\mathcal{T}_e$. If λ_n denotes the normalized length measure of $(T_n, n^{-1/2}d_{T_n}, u_0^n)$, observe that, for all $u_i^n \in T_n$, $i \in \{0,...,2n\}$,

$$\lambda_n([u_0^n, u_i^n]]) = n^{-1/2}d_{T_n}(u_0^n, u_i^n) = n^{-1/2}C_n(i).$$

The normalized length measure λ_n is naturally associated with a σ-finite measure λ_{C_n} on $(\{0,...,2n\}, n^{-1/2}d_{C_n}, 0)$, such that for all $i \in \{0,...,2n\}$,

$$\lambda_{C_n}((0,i]) = n^{-1/2}d_{C_n}(0,i) = n^{-1/2}C_n(i) = \lambda_n([u_0^n, u_i^n]]),$$

where d_{C_n} is defined similarly to (2.2) replacing g with C_n. Recall here that C_n is also a positive excursion with finite length $2n$. In a similar fashion, let λ_e be the unique σ-finite measure on $([0,1], d_e, 0)$, such that for each $t \in [0,1]$,

$$\lambda_e((0,t]) = d_e(0,t) = d_e(p_e(0), p_e(t)) = \lambda([p_e(0), p_e(t)]]),$$

where λ is the length measure of $\mathcal{T}_e$. It is a fact that the normalized length measure λ_n of the discrete tree T_n shifts the length of one edge to its endpoint that lies further away from the root u_0^n. Hence, for every $u_i^n, u_j^n \in T_n$, $i,j \in \{0,...,2n\}$, the sum and consequently the distorted distance in (5.38) between u_i^n and u_j^n can be rewritten as

$$n^{-1/2}r_n(u_i^n, u_j^n) = \int_{[u_i^n, u_j^n]]} e^{n^{-1/4}V_n(v)}\lambda_n(\mathrm{d}v) = \int_{[i,j]]} e^{n^{-1/4}V_n(u_k^n)}\lambda_{C_n}(\mathrm{d}k).$$

Similarly, the distorted distance $r_{\phi^{(1)}}$, see (5.39), between $p_e(s)$ and $p_e(t)$, for some $s,t \in [0,1]$, can be reexpressed as

$$r_\phi(p_e(s), p_e(t)) = \int_{[[p_e(s),p_e(t)]]} e^{\sigma_{\beta,\phi}\phi^{(1)}(v)} \lambda(\mathrm{d}v) = \int_s^t e^{\sigma_{\beta,\phi}\phi^{(1)}(p_e(r))} \lambda_e(\mathrm{d}r).$$

Hence, for $(i,s),(j,t) \in R_n$, we have that

$$\begin{aligned}
&\left| n^{-1/2} r_n(u_i^n, u_j^n) - r_\phi(p_e(s), p_e(t)) \right| \\
&= \left| \int_{[i,j]]} e^{n^{-1/4} V_n(u_k^n)} \lambda_{C_n}(\mathrm{d}k) - \int_s^t e^{\sigma_{\beta,\phi}\phi^{(1)}(p_e(r))} \lambda_e(\mathrm{d}r) \right| \\
&= \left| \int_s^t e^{n^{-1/4} V_n(u^n_{\lfloor 2nr \rfloor})} \lambda_{C_n}(\mathrm{d}r) - \int_s^t e^{\sigma_{\beta,\phi}\phi^{(1)}(p_e(r))} \lambda_e(\mathrm{d}r) \right|,
\end{aligned}$$

which is bounded above by

$$\begin{aligned}
&\leq \sup_{(i,t)\in R_n} |e^{n^{-1/4} V_n(u_i^n)} - e^{\sigma_{\beta,\phi}\phi^{(1)}(p_e(t))}| \cdot \lambda_{C_n}((\lfloor 2ns \rfloor, \lfloor 2nt \rfloor]) \\
&+ \left| \int_s^t e^{\sigma_{\beta,\phi}\phi^{(1)}(p_e(r))} \lambda_{C_n}(\mathrm{d}r) - \int_s^t e^{\sigma_{\beta,\phi}\phi^{(1)}(p_e(r))} \lambda_e(\mathrm{d}r) \right|. \qquad (5.41)
\end{aligned}$$

For each $s,t \in [0,1]$,

$$n^{-1/2} \lambda_{C_n}((\lfloor 2ns \rfloor, \lfloor 2nt \rfloor]) \to \lambda_e((s,t]),$$

as $n \to \infty$. Combining this with (5.36) yields that both terms in (5.41) converge to 0, uniformly in $s,t \in [0,1]$, as $n \to \infty$, and the part of the proof that shows that the distortion $\mathrm{dis}(R_n)$ of the correspondence converges to 0, is complete.

We now introduce what we call the distorted contour exploration of T_n. In essence, what it does is to collect a weight equal to $e^{-n^{-1/4} V_n(u_i^n)}$, $i \in \{0, ..., 2n\}$, whenever the directed edge connecting the parent of u_i^n to u_i^n is traversed in the canonical contour exploration of T_n. To be more precise, set

$$\tilde{C}_n(i) := \sum_{u \in [u_0^n, u_i^n]]} e^{-n^{-1/4} V_n(u)}, \qquad 0 < i < 2n.$$

By convention, let $\tilde{C}_n(0) = \tilde{C}_n(2n) := 0$. Extend $\tilde{C}_n$ by linear interpolation to non-integer times. Then, $(T_n, n^{-1/2}r_n, u_0^n)$ is a random real tree coded by $\tilde{C}_n$. The mass measure $\mu_{\tilde{C}_n}$ on T_n is defined as the image measure by the canonical projection $p_{\tilde{C}_n}$ of the Lebesgue measure on $[0, 2n]$. By definition, $(2n)^{-1}\mu_{\tilde{C}_n}(A) = \ell(\{t \in [0,1] : p_{\tilde{C}_n}(t) \in A\})$, for a Borel set A of $(T_n, n^{-1/2}r_n, u_0^n)$. The Prokhorov distance between $(2n)^{-1}\mu_{\tilde{C}_n}$ and $\nu_{\phi^{(1)}}$ is negligible since

$$d_{T_n}^P\left((2n)^{-1}\mu_{\tilde{C}_n}, (2n)^{-1}\nu_n\right) \leq (2n)^{-1},$$

recalling that ν_n is the stationary reversible measure of the weakly biased random walk, see (5.37). Towards proving that the Prokhorov distance between $(2n)^{-1}\mu_{\tilde{C}_n}$ and $\nu_{\phi^{(1)}}$ is negligible, we consult the proof of [2, Proposition 2.10]. There exists a common metric space (Z, d_Z), such that

$$d_Z^P\left((2n)^{-1}\mu_{\tilde{C}_n}, \nu_{\phi^{(1)}}\right) \leq \frac{1}{2}\mathrm{dis}(R_n) + |\mathrm{supp}(\tilde{C}_n) - \mathrm{supp}(\tilde{e})|.$$

Since the right-hand-side converges to 0 as $n \to \infty$, the desired result follows.

$\square$

The $\nu_{\phi^{(1)}}$-speed motion on $(\mathcal{T}_e, \sigma_T r_{\phi^{(1)}})$, which we coined the $\nu_{\phi^{(1)}}$-Brownian motion in a random Gaussian potential $\sigma_{\beta,\phi^{(1)}}\phi^{(1)}$ on the Brownian CRT, is a novel object that emerges as the annealed scaling limit of the weakly biased random walk $(X_m^n)_{m\geq 1}$ on T_n, with bias parameter $\beta^{n^{-1/4}}$, for some $\beta > 1$. To make this statement clear, we suppose that the random elements

$$\left((T_n, n^{-1/2}r_n, \varrho^n), (2n)^{-1}\nu_n, n^{-1/4}\phi_n, n^{-1/4}V_n\right)_{n\geq 1}$$

and

$$\left((\mathcal{T}_e, \sigma_T r_{\phi^{(1)}}, \varrho), \nu_{\phi^{(1)}}, \Sigma_\phi \phi, \sigma_{\beta,\phi}\phi^{(1)}\right)$$

are built on a probability space with probability measure $\mathbf{P}$. This is possible since the probability measure $\mathbb{M}_n$ on $C([0,1], \mathbb{R}_+) \times C([0,1], \mathbb{R}^d)$ such that the pair of the normalized discrete tours $(C_{(n)}, R_{(n)})$ is in its support, converges weakly as a probability measure to $\mathbb{M}$, a probability measure on $C([0,1], \mathbb{R}_+) \times C([0,1], \mathbb{R}^d)$ defined similarly in such a way that the resulting spatial tree $(\mathcal{T}_e, \phi)$ has marginal $\mathbb{M}$, see [64, Theorem 2]. Then, $\mathbf{P}$ is the probability measure of the probability space under which the aforementioned weak convergence holds almost-surely, which we can assume exists using Skorohod's representation theorem. The annealed laws

$\mathbb{P}^{\varrho^n}$ and $\mathbb{P}^{\varrho}$ of the weakly biased random walk $(X^n_m)_{m\geq 1}$ and the $\nu_{\phi^{(1)}}$-Brownian motion in a random Gaussian potential $\sigma_{\beta,\phi^{(1)}}\phi^{(1)}$ respectively, are obtained by integrating out the randomness of the state spaces with respect to $\mathbf{P}$.

Finally, we are able to state our result, as (5.12) and (5.13) are satisfied, and therefore so is Assumption 5. (5.13) simply follows from the fact that the spaces involved in the spatial Gromov-Hausdorff-vague convergence of Theorem 5.4.1 are compact, cf. the proof of (4.39).

Theorem 5.4.2. *Consider the weakly biased random walk $(X^n_m)_{m\geq 1}$ on T_n with bias parameter $\beta^{n^{-1/4}}$, for some $\beta > 1$. Then,*

$$\mathbb{P}^{\varrho^n}\left(\left(n^{-1/4}\phi_n(X^n_{n^{3/2}t})\right)_{t\geq 0}\in\cdot\right)\to\mathbb{P}^{\varrho}\left(\left(\Sigma_\phi\phi(X_{t\sigma_T^{-1}})\right)_{t\geq 0}\in\cdot\right),$$

weakly as probability measures on $D(\mathbb{R}_+,\mathbb{R}^d)$, where $\sigma_T > 0$ is a constant, Σ_ϕ is a positive definite $d\times d$-matrix, $(X_t)_{t\geq 0}$ is the $\nu_{\phi^{(1)}}$-Brownian motion in a random Gaussian potential $\phi^{(1)}$ on the Brownian CRT, $\phi^{(1)}$ is the first coordinate of a tree-indexed Gaussian process $(\phi(\sigma))_{\sigma\in\mathcal{T}_e}$ with $\mathbf{E}\phi(\sigma)=0$ and covariance structure

$$\mathrm{Cov}(\phi(\sigma),\phi(\sigma'))=d_e(\varrho,\sigma\wedge\sigma')I,$$

where I is the d-dimensional identity matrix, if $(\mathcal{T}_e,d_e)$ denotes the Brownian CRT, a real tree coded by a normalized Brownian excursion, endowed with its canonical metric (2.2).

5.5 Edge-reinforced random walk on large critical trees

Let $(\alpha^n_0(e))_{e\in E(T_n)}$ be a sequence of positive initial weights on $E(T_n)$, the set of edges of a critical Galton-Watson tree with finite variance for the aperiodic offspring distribution, the model that was fully described in Section 5.4. The edge-reinforced random walk (ERRW) on T_n, started from ϱ^n, is introduced as the discrete time process $Z=((Z^n_k)_{k\geq 1},\mathbf{P}^u_{\alpha_0},u\in T_n)$ with transition probabilities

$$\mathbf{P}_{\alpha_0}(Z^n_{k+1}=u|(Z^n_j)_{0\leq j\leq k})=\mathbb{1}_{\{u\sim Z^n_k\}}\frac{N^n_k(\{Z^n_k,u\})}{\sum_{u'\sim Z^n_k}N^n_k(\{Z^n_k,u'\})},$$

where for an edge $e \in E(T_n)$, $N_k^n(e) := \alpha_0^n(e) + \#\{0 \le j \le k-1 : \{Z_j^n, Z_{j+1}^n\} = e\}$. In other words, at time k, this walk jumps through a neighboring edge e with probability proportional to $N_k^n(e)$, which is initially equal to $\alpha_0^n(e)$ and then increases by 1 each time the edge e is crossed before time k. The initial weights we are going to be interested in choosing are

$$\alpha_0^n(e) = 2^{-1} n^{1/2}, \qquad e \in E(T_n), \tag{5.42}$$

so that the ratio of the initial weights over the shortest path metric, when rescaled by $n^{-1/2}$, is constant. The following theorem due to Sabot and Tarrès describes the ERRW as a mixture of Markovian random walks.

Theorem 5.5.1 (Sabot, Tarrès [93]). *Let $\alpha^n := (\alpha^n(e))_{e \in E(T_n)}$ independent random variables with $\alpha^n(e) \sim \Gamma(\alpha_0^n(e), 1)$. Let $(\omega^n(e_i(u)) : 0 \le i \le \xi(u))_{u \in T_n}$ be an independent family of independent random variables, that conditional on α^n, are distributed according to the density*

$$\sqrt{\frac{\alpha^n(e_i(u))}{\pi}} e^{-2\alpha^n(e_i(u)) \sinh\left(\frac{x}{2}\right)^2 + \frac{x}{2}} dx, \tag{5.43}$$

where $(e_i(u))_{i=0}^{\xi(u)} := (\{u, u_i\} : 0 \le i \le \xi(u))$. Define $\mathcal{U}^n := (\mathcal{U}^n(u))_{u \in T_n}$ by

$$\mathcal{U}^n(u) := \begin{cases} \sum_{e \in E_{\varrho^n, u}} \omega^n(e), & u \neq \varrho^n, \\ 0, & u = \varrho^n, \end{cases}$$

where $E_{\varrho^n, u}$ is the set of all edges contained in the unique path connecting ϱ^n and u. $\mathcal{U}^n$ is interpolated linearly along the edges. Consider the nearest neighbor random walk on T_n, started from ϱ^n, that conditional on $(\alpha^n, \mathcal{U}^n)$, moves from u to u_i with probability

$$\alpha^n(e_i(u)) e^{-(\mathcal{U}^n(u) + \mathcal{U}^n(u_i))}.$$

Then, under the annealed law it has the same distribution as the ERRW $(Z_k^n)_{k \ge 0}$.

As a consequence of the theorem above and (5.2), the potential $\mathcal{V}^n := (\mathcal{V}^n(u))_{u \in T_n}$ of the random walk in random environment $(\alpha^n, \mathcal{U}^n)$ has the following expression:

$$\mathcal{V}^n(u) = \begin{cases} \mathcal{U}^n(\vec{u}) + \mathcal{U}^n(u) + \log \alpha^n(\{\vec{u}, u\})^{-1}, & u \neq \varrho^n, \\ 0, & u = \varrho^n, \end{cases}$$

The aim of the following series of lemmas is to establish the distributional convergence of this potential and examine its limit. In what follows, it is useful to recall the correspondence R_n between T_n and $\mathcal{T}_e$ that was extensively used in the proof of Theorem 5.4.1.

Lemma 5.5.2. *Suppose that $(i, t) \in R_n$. Then,*

$$\sup_{t\in[0,1]} \left| \frac{1}{2} \sum_{e \in E_{u_0^n, u_i^n}} \alpha^n(e)^{-1} - d_e(p_e(0), p_e(t)) \right| \xrightarrow{P} 0,$$

as $n \to \infty$, where the convergence above is in probability.

Proof. Since $\alpha^n(e) \sim \Gamma(\alpha_0^n(e), 1)$, then $\alpha^n(e)^{-1}$ follows the inverse Gamma distribution with parameters $\alpha_0^n(e)$ and 1. For n large enough, by elementary properties of the Gamma distribution, we derive the following asymptotic behavior of the mean and variance of $\alpha^n(e)^{-1}$. Note that for n large, the expressions below are well-defined as $\alpha_0^n(e)$ diverges, see (5.42).

$$\mathbf{E}(\alpha^n(e)^{-1}) = (\alpha_0^n(e) - 1)^{-1} = O(\alpha_0^n(e)^{-1}) = O(n^{-1/2}),$$

$$\mathrm{Var}(\alpha^n(e)^{-1}) = (\alpha_0^n(e) - 1)^{-2}(\alpha_0^n(e) - 2)^{-1} = O(n^{-3/2}).$$

Using Kolmogorov's maximal inequality, for every $\eta > 0$,

$$\mathbf{P}\left(\sup_{t\in[0,1]} \frac{1}{2} \left| \sum_{e \in E_{u_0^n, u_i^n}} \left[\alpha^n(e)^{-1} - \mathbf{E}(\alpha^n(e)^{-1})\right] \right| > \eta \right) \leq \frac{\sum_{e \in E_{u_0^n, u_i^n}} \mathrm{Var}(\alpha^n(e)^{-1})}{4\eta^2}$$

$$= \frac{O(n^{-3/2} d_{T_n}(u_0^n, u_i^n))}{4\eta^2},$$

which goes to 0, as $n \to \infty$. This in turn yields the desired result just by noticing

that

$$\limsup_{n\to\infty}\left|\frac{1}{2}\sum_{e\in E_{u_0^n,u_i^n}}\mathbf{E}\left(\alpha^n(e)^{-1}\right)-d_e(p_e(0),p_e(t))\right|$$

$$\leq \limsup_{n\to\infty}|n^{-1/2}d_{T_n}(u_0^n,u_i^n)-d_e(p_e(0),p_e(t))|,$$

which is equal to 0, uniformly in $t\in[0,1]$.

$\square$

Lemma 5.5.3. *As $n\to\infty$, conditional on $(\alpha^n,\mathcal{U}^n)$,*

$$\left((T_n,n^{-1/2}d_{T_n},\mu_{T_n},\varrho^n),\mathcal{V}^n\right)\xrightarrow{(d)}\left((\mathcal{T}_e,\sigma_T d_e,\mu_{\mathcal{T}_e},\varrho),2\mathcal{U}\right),$$

in the spatial Gromov-Hausdorff-vague topology, where $\mathcal{U}:=(\mathcal{U}(u))_{u\in\mathcal{T}_e}$ is a process defined by

$$\mathcal{U}(u):=\sqrt{2}\phi(u)+d_e(\varrho,u),\qquad u\in\mathcal{T}_e, \tag{5.44}$$

where $(\phi(u))_{u\in\mathcal{T}_e}$ is a tree-indexed Gaussian process built on a probability space with probability measure $\mathbf{P}$, with $\mathbf{E}\phi(u)=0$ and $\mathrm{Cov}(\phi(u),\phi(u'))=d_e(\varrho,u\wedge u')$.

Proof. Note that $\mathcal{U}(p_e(t))-\mathcal{U}(p_e(s))$, $s,t\in[0,1]$ with $s\leq t$, is normally distributed with mean $d_e(p_e(s),p_e(t))$ and variance $2d_e(p_e(s),p_e(t))$, i.e.

$$N(d_e(p_e(s),p_e(t)),2d_e(p_e(s),p_e(t))).$$

Take $s,t\in[0,1]$ with $s\leq t$, such that $(i,s),(j,t)\in R_n$ and $\{u_i^n,u_j^n\}\in E(T_n)$. The conclusion of Lemma 5.5.2 gives that the total variation distance between

$$N((2\alpha^n(\{u_i^n,u_j^n\}))^{-1},\alpha^n(\{u_i^n,u_j^n\})^{-1})\text{ and }N(d_e(p_e(s),p_e(t)),2d_e(p_e(s),p_e(t)))$$

converges in probability to 0, as $n\to\infty$. The increment $\mathcal{U}^n(u_j^n)-\mathcal{U}^n(u_i^n)=\omega^n(\{u_i^n,u_j^n\})$ has its law, conditional on α^n, explicitly given in (5.43), and using a standard Kullback-Leibler divergence bound, see [96, (13)], the total variation distance between its law and that of $N((2\alpha^n(\{u_i^n,u_j^n\}))^{-1},\alpha^n(\{u_i^n,u_j^n\})^{-1})$ is

$$O(\alpha^n(\{u_i^n,u_j^n\})^{-1}).$$

Therefore, the total variation distance between the distribution of $\omega^n(\{u_i^n, u_j^n\})$ and that of $\mathcal{U}(p_e(t)) - \mathcal{U}(p_e(s))$ is of the same order as above. Again, due to Lemma 5.5.2, the fact that $|t-s| < n^{-1}$ for those $(i,s), (j,t) \in R_n$ with $\{u_i^n, u_j^n\} \in E(T_n)$ and the of e, almost-surely, we deduce that $\alpha^n(\{u_i^n, u_j^n\})^{-1}$ converges to 0 in probability, as $n \to \infty$. As a consequence, $(\mathcal{U}^n(u_i^n))_{t\in[0,1]}$ converges in law, as $n \to \infty$, to $(\mathcal{U}(p_e(t)))_{t\in[0,1]}$.

□

When ν_n and r_n are defined similarly to (5.37) and (5.38) respectively, with the potential of the particular RWRE studied in Section 5.4 replaced by $\mathcal{V}^n$, the proof of Theorem 5.4.1 remains intact. In our context (see (8) in [78] for details) the process $(\phi(u))_{u\in\mathcal{T}_e}$ has a continuous modification, therefore there exists a **P**-a.s. continuous modification of $\mathcal{U}$. The scaling limit of the ERRW on T_n with initial weights as in (5.42) is described as the $\nu_{\mathcal{U}}$-speed motion on $(\mathcal{T}_e, \sigma_T r_{\mathcal{U}}, \varrho)$, where

$$r_{\mathcal{U}}(u_1, u_2) := \int_{[[u_1,u_2]]} \exp(2\mathcal{U}(v))\lambda(\mathrm{d}v),$$

for every $u_1, u_2 \in \mathcal{T}_e$ and $\nu_{\mathcal{U}}$ is the mass measure on $\mathcal{T}_e$ defined as the image measure by the canonical projection $p_{\hat{e}}$ of the Lebesgue measure on $[0,1]$, see (2.4), where

$$\hat{e} := \left(\int_{[[p_e(0),p_e(t)]]} \exp(-2\mathcal{U}(v))\lambda(\mathrm{d}v) : 0 \le t \le 1 \right).$$

Theorem 5.5.4. *Consider the ERRW $(Z_k^n)_{k\ge1}$ on T_n, started at its root ϱ^n, with initial weights given by $\alpha_0^n(e) = 2^{-1}n^{1/2}$, $e \in E(T_n)$. Then, there exists a common metric space (Z, d_Z) onto which we can isometrically embed (T_n, r_n), $n \ge 1$ and $(\mathcal{T}_e, r_{\mathcal{U}})$, such that*

$$\mathbf{P}_{\alpha_0}^{\varrho^n}\left((n^{-1/2} Z^n_{n^{3/2}t})_{t\in[0,1]} \in \cdot\right) \to \mathbf{P}^{\varrho}\left((Z_{t\sigma_T^{-1}})_{t\in[0,1]} \in \cdot\right),$$

weakly as probability measures on $D(\mathbb{R}_+, Z)$, where $\sigma_T > 0$ is a constant, $(Z_t)_{t\ge0}$ is the $\nu_{\mathcal{U}}$-Brownian motion in a random potential $2(\phi(u) + d_e(\varrho,u))_{u\in\mathcal{T}_e}$ on the Brownian CRT, started at ϱ, ϕ and d_e are the same as in the statement of Theorem 5.4.2.

We emphasize that choosing T_n to be a critical Galton-Watson tree with finite variance for the aperiodic offspring distribution is justified by its distributional convergence as a metric measure space, and more importantly by the convergence of its contour function. Therefore, it is of no surprise that the theorem above is expected to hold for the ERRW on random ordered trees that possess these properties, such as a size-conditioned critical Galton-Watson tree, whose aperiodic offspring distribution lies in the domain of attraction of a stable law of index $\alpha \in (1, 2]$. It was shown by Duquesne [47], (see also [76]) that, properly rescaled, its contour function converges weakly to a normalized excursion of the continuous height function associated with the α-stable continuous-state branching process, which encodes the α-stable Lévy tree, a generalization of the Brownian CRT in the case $\alpha = 2$ (for definitions, see the references mentioned above).

Chapter 6

Future plans and open problems

I propose the following research projects to explore aging properties of the diffusions in random potential presented in the previous chapter.

1. The one-dimensional diffusion $(X_t)_{t\geq 0}$ in a random Wiener potential W considered by Brox in [28] exhibits some interesting features. Among those, there exists a non-trivial measurable function b_1 such that $(X_t)_{t\geq 0}$ converges very slowly to b_1, which is the so-called subdiffusivity, see the statement in (5.19) and cf. (5.16). This result is a consequence of a localization phenomenon that occurs, trapping the diffusion in some valleys of its potential, and was extended to a wide class of random environments [66]. Also, see [75] for a limit theorem for the shape of the full trajectory of a multi-dimensional diffusion in a self-similar random potential.

   We have a corresponding notion of a valley of the potential of the Brownian motion on $(\mathcal{T}, r_\phi)$, where $\mathcal{T}$ is the SSCRT, a continuum random sin-tree coded by left and right height processes that are two independent three-dimensional Bessel processes. Furthermore,

   $$r_\phi(u_1, u_2) = \int_{[[u_1,u_2]]} e^{\sigma_{\beta,\phi}\phi(v)}\lambda(\mathrm{d}v), \qquad u_1, u_2 \in \mathcal{T}$$

   is a metric on $\mathcal{T}$. Here, recall from (5.39) in Theorem 5.4.1 that $[[u_1, u_2]]$ is the unique arc connecting u_1 and u_2, $\sigma_{\beta,\phi}$ is a positive constant depending on β and Σ_ϕ, and λ denotes the length measure of the SSCRT.

   We call a valley or ℓ-valley a triple (L_ℓ, b_ℓ, h_ℓ), where L_ℓ is the sub-level domain of ϕ restricted to the closure $\mathcal{T}^{(\ell)}$ of the ball in $\mathcal{T}$ of radius ℓ centred at the root,

$$b_\ell := \arg_{x \in L_\ell} \min \phi(x)$$

is the base of the valley and, given ϕ's continuity,

$$h_\ell := \ell - \phi(b_\ell)$$

denotes its height, a term justified by noticing that $\sup_{x \in \partial L_\ell} \phi(x) = \ell$. We assume that the following hold with high probability as $\ell \to \infty$ and $\varepsilon \to 0$.

- $[[\varrho, b_\ell]] \subseteq L_{\ell(1-2\varepsilon)}$ and b_ℓ is the unique minimum for ϕ on $L_{\ell(1+\varepsilon)}$.
- the smallest number m such that there exists a set $A = \{z_1, ..., z_m\}$ with $r_\phi(x, z_i) \geq e^\ell/4$ for each i, such that any path from x to L_ℓ^c must pass through A, is of order $O(1)$.
- $\inf\{\phi(x) - \phi(b_\ell) : x \in V_{\ell(1+\varepsilon)} \setminus B_{d_\mathcal{T}}(b_\ell, \varepsilon\ell)\} \geq \varepsilon\ell$.
- $b_{\ell^{-1}(t)}$ is continuous at t, where $t \mapsto \ell^{-1}(t) := \inf\{\ell' : h_{\ell'} > \log t\}$ is the right-continuous inverse of e^h.

Then, under mild assumptions on the volume growth of L_ℓ for ℓ large enough, we claim that, for every $\delta > 0$,

$$\mathbb{P}^\varrho(d_\mathcal{T}(X_t, b_{\ell^{-1}(t)}) > \delta\ell^{-1}(t)) \xrightarrow{t\to\infty} 0, \tag{6.1}$$

where $\mathbb{P}^\varrho$ denotes the annealed law of $(X_t)_{t\geq 0}$ started from ϱ with respect to the law of the random environment. Therefore, identifying the unique b_ℓ and verifying all the aforementioned assumptions yields the limit theorem above.

2. A precursor to this problem allows one to get a glance and further conjecture on the precice nature of (6.1). The scaling limit of the ERRW on critical Galton-Watson trees conditioned to survive (Kesten [70] showed that it is possible to make sense of conditioning them to survive or 'grow to infinity') is a Brownian motion in a random Gaussian potential with a drift given by

$$\phi(u) + d_\mathcal{T}(\varrho, u), \tag{6.2}$$

where $\mathcal{T}$ is the SSCRT. Due to modulus of continuity properties of ϕ (see

[49, Theorem 6.4]), the decisive term in (6.2) regarding the occurence of a localization phenomenon is the drift $d_{\mathcal{T}}(\varrho, \cdot)$, which is an artefact of the self-reinforcement. Therefore, the sub-level domain L_ℓ is asymptotically $\mathcal{T}^{(\ell)}$, a toy model for which our four assumptions are expected to hold with $b_\ell = \varrho$.

Conjecture 1. *Let $(X_t)_{t\geq 0}$ be the Brownian motion in a random Gaussian potential with a drift given by* (6.2)*. For each $\delta > 0$,*

$$\lim_{t\to\infty} \mathbb{P}^{\varrho}\left(\frac{d_{\mathcal{T}}(\varrho, X_t)}{(\log t)} > \delta\right) = 0.$$

A stronger statement was confirmed to be valid in [83], where the large time behavior of the continuous space limit of the ERRW on $2^{-n}\mathbb{Z}$ was examined.

3. Consider a Galton-Watson tree T, which is a branching process with i.i.d. offsprings that are distributed as a random variable ξ with mean 1, $\sigma_\xi^2 = \mathrm{Var}(\xi) \in (0, \infty)$ and $\mathbf{E}(e^{\lambda\xi}) < \infty$, for some $\lambda > 0$. Given a realization of T, the lattice branching random walk on T assigns a spatial location $\phi_T(u) \in \mathbb{Z}^d$, for each $u \in T$. First, by setting the spatial location of the root to be the origin of the d-dimensional lattice. If $(y(e))_{e\in E(T)}$ are i.i.d. according to the step distribution of a simple random walk, then $\phi_T(u)$ is the sum of values $y(e)$ over the set of all edges contained in the unique path that connects u to the root. The couple (T, ϕ_T) is called the critical lattice branching random walk, and can be viewed as an embedded subgraph of $\mathbb{Z}^d$. Notice that it is not necessarily a tree. We propose to study the weakly biased random walk $(X^n_m)_{m\geq 1}$ with bias parameter $\beta^{n^{-1/4}}$, for some $\beta > 1$, started from 0, on the trace of the critical lattice branching random walk (T_n, ϕ_n), where T_n is a Galton-Watson tree conditioned to have size n.

Conjecture 2. *In high dimensions ($d > 14$ in the case where there is no bias and we end up with the simple random walk on the trace of critical lattice branching random walk [17])*

$$\left(n^{-1/4} X^n_{n^{3/2}t}\right)_{t\geq 0}$$

converges to the Brownian motion in a random Gaussian potential on the ISE. The convergence is annealed and takes place in the uniform topology over compact sets.

The process considered in Theorem 5.4.2 essentially differs from the considered in the conjecture above. More specifically, $(\phi_n(X_m^n))_{m\geq 1}$ is a biased random walk on a tree, which is then embedded. Recall that in Section 5.4, the step distribution according to which $(y(e))_{e\in E(T_n)}$ is distributed was assumed to be continuous in $\mathbb{R}^d$ with fourth order polynomial tail decay. In this open problem $(X_m^n)_{m\geq 1}$ is a biased random walk on a graph that contains cycles.

Appendix A

Proposition A.0.1. *Let* (G, μ^G) *be a weighted graph, with edges weighted according to* $(1/\mu^G_{xy})_{\{x,y\}\in E(G)}$*. Let* d_G *and* R_G *be the weighted shortest path distance and the resistance metric respectively. Then,* $d_G \geq R_G$*. If* G *is a graph tree, i.e. there is a unique path between any two vertices in the graph, then* $d_G \equiv R_G$*.*

Proof. Let $a, b \in V(G)$ and π be a path that connects a and b. For $f : V(G) \to \mathbb{R}$ that satisfies $f(a) = 0$ and $f(b) = 1$, applying the Cauchy-Schwarz inequality gives

$$\left(\sum_{\{x,y\}\in\pi} (\mu^G_{xy})^{-1}\right)\left(\sum_{\{x,y\}\in\pi} (f(x) - f(y))^2 \mu^G_{xy}\right) \geq \left(\sum_{\{x,y\}\in\pi} |f(x) - f(y)|\right)^2$$

Since $f(a) = 0$, $f(b) = 1$ and π is a path that connects a,b, the right-hand side of the inequality above is bounded below by $|f(a) - f(b)|^2 = 1$. Hence,

$$\mathcal{E}_G(f, f) \geq \sum_{\{x,y\}\in\pi} (f(x) - f(y))^2 \mu^G_{xy} \geq \left(\sum_{\{x,y\}\in\pi} (\mu^G_{xy})^{-1}\right)^{-1},$$

where $\sum_{\{x,y\}\in\pi} (\mu^G_{xy})^{-1}$ denotes the weighted distance between a and b on π. Recalling the definition of the resistance metric on G, this implies $R_G(a, b) \leq \sum_{\{x,y\}\in\pi} (\mu^G_{xy})^{-1}$, which proves the desired inequality.

It is easy to see that all the inequalities in the proof hold with equality if there exists a unique path connecting a and b.

∎

Appendix B

Let $((K, R, \varrho), \pi, X) \in \mathbb{K}$ and B be a non-empty closed subset of K. By [74, Theorem 4.1], there exists a unique function $g_B : K \times K \to \mathbb{R}$ such that, for every $x \in K$, $g_B(x, \cdot) \in \mathcal{K}$ and

$$\mathcal{E}(g_B(x, \cdot), f) = f(x), \tag{B.1}$$

for every $f \in \{f \in \mathcal{K} : f|_B = 0\}$. As part of [74, Theorem 4.1], g_B satisfies

$$0 \leq g_B(x, y) = g_B(y, x) \leq g_B(x, x) = R(x, B). \tag{B.2}$$

Furthermore, by [74, Theorem 10.4], the transition density $(p_t^{K \setminus B}(x, y))_{x,y \in K, t>0}$ of the corresponding Hunt process X^B, which is the process X with the killing condition on hitting B, exists and is continuous on $K \times K \times (0, \infty)$:

$$g_B(x, y) = \int_0^\infty p_t^{K \setminus B}(x, y) dt, \qquad \forall x, y \in K.$$

This readily implies

$$\mathbf{E}_y \left(\int_0^{\tau_B} f(X_t) dt \right) = \int_K g_B(y, z) f(z) \pi(dz), \qquad \forall y \in K, \tag{B.3}$$

for any measurable function $f : K \to \mathbb{R}_+$, where τ_B is the hitting time of B.

Proposition B.0.1. *If* $((K, R, \varrho), \pi, X) \in \mathbb{K}$, *then*

$$\mathbf{E}_x \tau_y + \mathbf{E}_y \tau_x = R(x, y) \pi(K), \qquad \forall x, y \in K,$$

where τ_z *is the hitting time of* z *by* X.

Proof. Fix $x, y \in K$. As in (B.1), there exists a function $g_{\{x\}} : K \times K \to \mathbb{R}$ such that, for every $y \in K$, $g_{\{x\}}(y,\cdot) \in \mathcal{K}$ and

$$\mathcal{E}(g_{\{x\}}(y,\cdot), f) = f(y),$$

for every $f \in \{f \in \mathcal{K} : f(x) = 0\}$. We deduce that

$$\mathcal{E}(g_{\{x\}}(y,\cdot) + g_{\{y\}}(x,\cdot), f) = \mathcal{E}(g_{\{x\}}(y,\cdot), f - f(x)) + \mathcal{E}(g_{\{y\}}(x,\cdot), f - f(y)) = 0,$$

for every $f \in \mathcal{K}$. It follows that $g_{\{x\}}(y,\cdot) + g_{\{y\}}(x,\cdot)$ is a constant. So,

$$g_{\{x\}}(y,\cdot) + g_{\{y\}}(x,\cdot) = g_{\{x\}}(y,x) + g_{\{y\}}(x,x) = R(x,y), \tag{B.4}$$

where we made use of (B.2). To conclude, $g_{\{x\}}(y,\cdot)$ is the occupation density for X, started at y with the killing condition on hitting x (cf. (B.3)), and so by symmetry and (B.4), we have that

$$\begin{aligned}
\mathbf{E}_x\tau_y + \mathbf{E}_y\tau_x &= \int_K g_{\{y\}}(x,z)\pi(dz) + \int_K g_{\{x\}}(y,z)\pi(dz) \\
&= \int_K (g_{\{x\}}(y,z) + g_{\{y\}}(x,z))\pi(dz) = \int_K R(x,y)\pi(dz) = R(x,y)\pi(K).
\end{aligned}$$

□

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